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Rough JU-Algebras in Quantales Via Core Successor Classes Based on Fuzzy Relations

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ABSTRACT

Fuzzy sets (FSs), rough sets (RSs), quantales, and core successor classes (CSCs) are mathematical tools that are applicable in real-world situations with imprecise and ambiguous data. In this research, we propose the novel idea of RSs on JU-algebra in quantales that are based on CSCs under fuzzy relations (FRs). The following article contains a notion that has never been introduced before. This merges the JU-algebras and RSs in quantales to provide a new approach for reducing uncertainty in various fields. We develop a mathematical model for approximate substructures on JU-algebras via CSCs, based on which options can be evaluated in a situation where the information is imprecise. The proposed approach is a recommended technique for coping with ambiguity and uncertainty in multiple fields. Applications in different fields are described, and theoretical foundations of rough JU-algebras are explored. Our findings demonstrate the effectiveness of the suggested strategy in handling ambiguity.

Keywords: Rough Sets, Fuzzy Sets, JU-Algebra, Quantales, Core Successor Classes

INTRODUCTION

Motivation and Research Gap

Pawlak tackled uncertain knowledge in the fields of AI information systems, and cognitive sciences in areas including ML, knowledge acquisition, and decision analysis at the start of the 1980s, among others, by utilizing the concept of RSs and its approximations (LAs and UAs) spaces in his papers [1-5]. The relationship between RSs and algebraic systems (structures) was formed via investigation and creativity. As a result, several concepts were introduced by various authors.

Acronyms	Representation	Acronyms	Representation
$\circledast$	Binary operation	CPFR	Compatible preorder fuzzy relation
UA	Upper Approximation	RSs	Rough sets
LA	Lower Approximation	RST	Rough set theory
FR	Fuzzy relation	CFR	Complete fuzzy relation
SCs	Successor classes	SFR	Serial fuzzy relation
CSCs	Core successor classes	DM	Decision-Making

Table 1: List of Acronyms

Numerous arbitrary binary relations, motivated by Pawlak's rough set and induced by an equivalence relation, have been used to study vast concepts. $SN\pi(\ ) = \{\ ' \in X: (x,\ ') \in \pi\}$ indicates a successor neighborhood of u produced by a binary relation on a universal set U in which u is an element of U. Yao, suggested roughness models based on binary-induced successor neighborhoods [6]. A framework based on binary relations was given for analyzing neighborhood systems and RS approximations, and he explored the features of neighborhood and approximation operators and their relationships. If u is an element in X, then $SCN\pi(\ ) = \{\ ' \in X: SN\pi(\ ) = SN\pi(\ )\}$ denotes the core successor neighborhood generated by a binary relation on a universal set X. RSs were proposed as core successor neighborhoods resulting from binary relations by Mareay [7]. If the property of being an equivalence relation is present in a binary relation on a universal set, then Mareay's RSs and Yao's RSs are extensions of Pawlak's RSs.

In what may be regarded as the notion of roughness in classical algebras, Kuroki, presented roughness in semigroups and its ideals [8]. The concept of rough groups and rough subgroups was later established by Biswas and Nanda [9]. Li and Yin, proposed ϑ-LA and T-UA fuzzy rough approximation operators on a semigroup, whereas Qi and Liu, investigated the ideas of rough approximation in Boolean algebras [10-11]. It was Davvaz, who first proposed the idea of roughness in rings [12].

Rough set theory (RST) was studied by Ameri, in relation to hyper BCK-algebras and by Dudek, in relation to hyper BCI-algebras [13-14]. Roughness concepts in BCC algebras have also been studied by Jun [15]. The origin of FSs goes back to 1965, when they were first developed by Zadeh [16]. Since then, it has been the subject of numerous studies by many authors in several scientific, technical, and engineering fields. Fuzziness, and forms of logical algebras are recent developments and topics of interest for many writers and researchers [17-20].

A number of authors are among them. In recent years, Rough fuzzy ideals in BCK/BCI algebras were investigated by Ahn, and others. Understanding the many kinds and orientations of applied algebras has been largely based on logical and classical algebras [21]. This is particularly true for artificial intelligence related computer systems that simulate how a person would handle the certainty and uncertainty of information using logical approaches [22]. Using these notions to tackle challenges is made easy by these tactics. BCI/BCK algebras and BL-algebras are two important categories of fundamental logical algebras [23].

Perfect BL-algebras, local BL-algebras, SBL-algebras, and all of its variations have been the subject of several investigations. Whether or not they have hyperstructures, these ideas relate to roughness and softness in an intriguing way. Impan introduced the roughness notion lately, and it has been used to UA-algebras [24]. UA-algebras are subject to a number of traditional theorems from RST, and related findings have been analyzed using this idea. A UA-algebra's rough ideal characteristics have been demonstrated using lower approximations (LAs) and upper approximations (UAs). It is demonstrated that a strong UA-ideal is once again a strong ideal in terms of a UA-algebra's UAs and Las.

Motivation and Research Gap

The motivation for this study is to link the fuzzy relational uncertainty with algebraic roughness closer. The RSs approach is successfully applied in various algebraic systems, but the FR induced CSCs and JU-algebraic systems have not been systematically studied.

The primary research gaps, as identified in this study are:

Approximation to the JU-Algebra Using the CSC Approach

The present research deals with the problem of constructing rough JU-substructures and analyzing their algebraic attributes of CSCs based on FRs and its RS approaches, which were developed for a few logical algebras.

Preservation Of JU-Algebraic Structures

It is not sufficient to define lower and upper approximations on a JU-algebra. It also needs to be determined whether they are compatible with the algebraic operation. Thus, the aim of this study is to investigate the preservation of JU-subalgebras, JU-ideals, weak JU-ideals and strong JU-ideals under the proposed approximations.

Approximation by Fuzzy Relations and Dual-Universe

Fuzzy relationships allow to directly address the consideration of uncertain relationships between the two universes in the approximation process. CSCs give the corresponding information granules to build the lower and upper approximations.

Extension to Quantales

The framework is then extended to quantales, to see if the algebraic structure proposed above is compatible with a richer complete-lattice based algebraic structure. The quantale extension is concerned with rough JU-ideals, weak JU-ideals and strong JU-ideals.

For that reason, the main research question of this paper is:

Under what conditions do FR induced CSC-based lower and upper approximations preserve important JU-algebraic structures, and how can this framework be extended to quantales?

This question is different from transferring an already known rough-set definition to another algebraic structure. Rather, the mathematical relationships and interactions between fuzzy relational granulation, the CSC-based approximation operators and the algebraic preservation properties are emphasized.

Key Contributions

Our novel contributions are:

CSC-Based Fuzzy Rough Approximations: We use the CSCs induced by FRs to build the lower and upper approximations, instead of classical equivalence relations.

Interaction with JU-Structures: We investigate that these approximations preserve JU-subalgebras, JU-ideals, weak JU-ideals and strong JU-ideals.

Extension to Quantales: The framework is extended to quantales and results on rough JU-ideals and variants are given.

Dual-Universe Fuzzy Relational Model: The model includes fuzzy relations and CSCs in order to represent the uncertainty existing between two related universes.

New Structural Preservation Results: New results on interaction between CSC-based approximations and algebraic operations and ideal structures.

Summary of The Work

This is a summary of the paper organization. The preliminary result and some of the basic definitions necessary for the subsequent analysis are given in Section 2. The principal theoretical results are extended to quantales, the development of the CSC-based rough JU-substructures under fuzzy relations is carried out in Section 3. Section 4 discusses how the proposed framework compares with the current approaches. Finally, the paper ends with Section 5 that summarizes the findings and provides some directions for future research.

MATERIAL AND METHODS

JU-algebras, JU-subalgebras, and JU-ideals are some of the fundamental definitions that will be covered in this part, along with some associated fundamental findings and examples. These discoveries will support our efforts.

Definition 2.1 Error! Reference source not found.

When any $\mathfrak{a}, \mathfrak{b}, \ \in$, fulfill the following identities, $\mathbb{JU}$-algebra is defined as an algebra (X,⊛,ꓔ) with a single binary operation ⊛.

$$(\mathbb{JU}_1)\quad (\mathfrak{b} \circledast \mathfrak{c})[(\mathfrak{c} \circledast \mathfrak{a}) \circledast (\mathfrak{b} \circledast \mathfrak{a})] = \daleth,$$

$$(\mathbb{JU}_2)\quad \daleth \circledast \mathfrak{a} = \mathfrak{a},$$

$$(\mathbb{JU}_3)\quad \mathfrak{a} \circledast \mathfrak{b} = \mathfrak{b} \circledast \mathfrak{a} = \daleth \Rightarrow \mathfrak{a} = \mathfrak{b}.$$

The fixed element of is the constantꓔ. For convenience, we express a$\mathbb{JU}$-algebra by writing rather than (,⊛, ꓔ).

Example 2.1: Error! Reference source not found. In the case of ={0, $\mathfrak{a}$, $\mathfrak{b}$, , }, where⊛ is established by the subsequent **Error! Reference source not found.**

⊛					
	0				
	0	0			
	0		0		
	0	0		0	
	0	0	0	0	0

Table 2: The set defines a binary operation⊛. The fact that is $\mathbb{JU}$-algebra is obvious. **Definition 2.2** Error! Reference source not found.

Fuzzy set (FS) of is the mappingη: → [0,1] when is a nonempty universal set.

Definition 2.3 Error! Reference source not found.

Fuzzy relation (FR) from to is defined as mapping η: × → [0,1] given two nonempty universal sets and . A mapping on is termed FR if it is η: × → [0,1]. An FR represented as a matrix is represented by

$$\begin{pmatrix} \eta_{11} & \eta_{12} \cdots & \eta_{1m} \\ \eta_{21} & \eta_{22} \cdots & \eta_{2m} \\ \vdots & \vdots & \vdots \\ \eta_{n1} & \eta_{n2} & \eta_{nm} \end{pmatrix}$$

Definition 2.4 Error! Reference source not found.

Assume that represents a FR between to . If is in and for every in , ($\mathfrak{b}$,) = 1 then η is serial FR (SFR).

Definition 2.5 Error! Reference source not found.

Let be a FR on , then:

1. It is said to be a reflexive FR if (,) = 1, ∀ q∈ ;
2. It is said to as a symmetric FR if ($_1$, $_2$) = ($_2$, $_1$), ∀ $_1$, $_2$ ∈ ;
3. Transitive FR if ($_1$, $_2$) ≥∨ $_3$∈ (($_1$, $_3$) ∧ ($_3$, $_2$)) ∀ $_1$, $_2$ ∈ .

Definition 2.6 Error! Reference source not found.

is referred to as similarity FR if it is reflexive, symmetric, and transitive FR.

Definition 2.7 Error! Reference source not found.

If ⊛ ∈ , ∀ 𝔞, ∈ , then ⊆ is 𝕁𝕌-subalgebra of .

Definition 2.8 Error! Reference source not found.

If ϕ ≠ 𝓙 ⊆ fulfills the following specifications, it is referred to as the𝕁𝕌-ideal of :

(1) 0 ∈ 𝓙,

(2) ⊛ (⊛) ∈ 𝓙, ∈ 𝓙 ⇒ ⊛ ∈ 𝓙, ∀ 𝔞, 𝔟, ∈ .

Example 2.2 Error! Reference source not found.: The **Error! Reference source not found.**3 that follows defines = {0, 𝔞, 𝔟, 𝔠, 𝔡, 𝔢} and ⊛.

⊛						
	0					
	0	0				
	0	0	0			
	0	0	0	0		
	0	0	0		0	
	0	0	0	0	0	0

Table 3: The set defines a binary operation ⊛.

It is obvious that (𝒳,⊛,ך) is a𝕁𝕌-algebra. It is easy to demonstrate that = {0, 𝔞} and = {0, 𝔞, 𝔟, 𝔠, 𝔡} are 𝕁𝕌-ideals of .

Definition 2.9 Error! Reference source not found.

Given a 𝕁𝕌-algebra and 0∈ 𝓙, let ϕ ≠ 𝓙 ⊆ .

Then

𝓙 is said to be a weak 𝕁𝕌-ideal of if ⊛ ∈ 𝓙 and ∈ 𝓙 ⇒ ∈ 𝓙, ∀ 𝔞, ∈ 𝒳;

If ⊛ ∩ 𝓙 ≠ ϕ and ∈ 𝓙 ⇒ ∈ 𝓙, ∀ 𝔞, ∈ , then 𝓙 is said to be a strong𝕁𝕌-ideal of .

Definition 2.10 Error! Reference source not found.

Let ς be in the interval [0,1] and be a FR from to , then for each in :

SC_η(; ς): = { ∈ : η(,) ≥ ς} (1)

It is known that a successor class (SC) of related to ς-level under η.

Definition 2.11 Error! Reference source not found.

Assume $\varsigma \in [0,1]$ and that is a FR from to , then for $_1 \in$:

$$CS\ _\eta(\ _1;\varsigma) := \{\ _2 \in\ : SC_\eta(\ _1;\varsigma) = SC_\eta(\ _2;\varsigma)\} \qquad (2)$$

is referred to as a core successor class (CSC) of $_1$ associated with ς-level under . For any in ,

we represent the collection of $CS\ _\eta(\ ;\varsigma)$ by $SC_\eta(\ ;\varsigma)$.

Proposition 2.1 Error! Reference source not found.

Let ς be in the interval [0,1] and η be a FR from to .

Then the properties listed below holds:

$\in CS\ _\eta(\ ;\varsigma)$ for all $\in$

For all $_1,\ _2 \in\ ,\ _2 \in CS\ _\eta(\ _1;\varsigma) \Leftrightarrow CS\ _\eta(\ _1;\varsigma) = CS\ _\eta(\ _2;\varsigma)$.

Definition 2.12 Error! Reference source not found.

Let ς be in the interval [0,1] and η be a FR from to . A triple $(\ ,\ ,\ SC_\eta(\ ;\varsigma))$ is referred to as an approximation space (AS) based on $SC_\eta(\ ;\varsigma)$. If $=$, then $(\ ,\ ,\ SC_\eta(\ ;\varsigma))$ can be written as $(\ ,\ SC_\eta(\ ;\varsigma))$.

Definition 2.13 Error! Reference source not found.

Let $(\ ,\ ,\ SC_\eta(\ ;\varsigma))$ be an $SC_\eta(\ ;\varsigma)$-AS and $\emptyset \neq\ \subseteq$, then we define UA of in $(\ ,\ ,\ SC_\eta(\ ;\varsigma))$ as:

$$\overline{\eta}(\ ;\varsigma) := \{\ \in\ : CS\ _\eta(\ ;\varsigma) \cap\ \neq \emptyset\}, \qquad (3)$$

and LA of in $(\ ,\ ,\ SC_\eta(\ ;\varsigma))$ as:

$$\underline{\eta}(\ ;\varsigma) := \{\ \in\ : CS\ _\eta(\ ;\varsigma) \subseteq\ \}, \qquad (4)$$

If $\overline{\eta}(\ ;\varsigma) \neq \underline{\eta}(\ ;\varsigma)$, Then $\eta\ (\ ;\varsigma) := (\overline{\eta}(\ ;\varsigma), \underline{\eta}(\ ;\varsigma))$ is known as a rough substructure of in $(\ ,\ ,\ SC_\eta(\ ;\varsigma))$.

Definition 2.14 Error! Reference source not found.

Let $(\ ,\ ,\ SC_\eta(\ ;\varsigma))$ be an $SC_\eta(\ ;\varsigma)$-AS and $\emptyset \neq\ \subseteq$. $\overline{\eta}(\ ;\varsigma)$ is called a nonempty $SC_\eta(\ ;\varsigma)$-UA of in $(\ ,\ ,\ SC_\eta(\ ;\varsigma))$ if $\overline{\eta}(\ ;\varsigma)$ is nonempty subset of .

$\underline{\eta}(\ ;\varsigma)$ is called a nonempty $SC_\eta(\ ;\varsigma)$-LA of in $(\ ,\ ,\ SC_\eta(\ ;\varsigma))$if $\underline{\eta}(\ ;\varsigma)$ is nonempty subset of . $\eta\ (\ ;\varsigma)$ is referred to as a nonempty $SC_\eta(\ ;\varsigma)$-rough set in $(\ ,\ ,\ SC_\eta(\ ;\varsigma))$ if $\overline{\eta}(\ ;\varsigma)$ is a nonempty $SC_\eta(\ ;\varsigma)$-UA and $\underline{\eta}(\ ;\varsigma)$ is a nonempty $SC_\eta(\ ;\varsigma)$-LA.

Definition 2.15 Error! Reference source not found.

We define FR η on . Then η is said to be compatible if $\forall\ _1,\ _2,\ _3,\ _4 \in$, we have

$$\eta(\mathfrak{b}_1 \circledast\ _3,\ _2 \circledast\ _4) \geq \eta(\mathfrak{b}_1,\ _2) \wedge \eta(q_3,\ _4) \qquad (5)$$

Definition 2.16 Error! Reference source not found.

Let η be a FR on . If η is reflexive, transitive and compatible then η is called a compatible preorder FR (CPFR). If η is a CPFR then $\mathcal{SC}_\eta($;ς)-AS (, $\mathcal{SC}_\eta($;ς)) is called an $\mathcal{SC}_\eta($;ς)-AS of CPFR.

Definition 2.17 Error! Reference source not found.

Assume that a nonempty set $\dot{\mathbb{Q}}$ is a complete lattice and that a binary operation ⊛ satisfies the assertion that ∀ , $_j$ ∈ $\dot{\mathbb{Q}}$:

$$\circledast (\vee_{\in} \ ) = \vee_{\in} (\ \circledast \ ),$$

$$(\vee_{\in} \ ) \circledast \ = \vee_{\in} (\ \circledast \ ).$$

It is then known as quantale. Suppose that $_j$, , ⊆$\dot{\mathbb{Q}}$ Following that, we define binary operations and arbitrary joins as

$$\vee \ = \{ {}_1 \vee \ | \ {}_1 \in \ , {}_2 \in \ \}, \quad (6)$$

$$\circledast \ = \{ {}_1 \circledast \ {}_2 | \ {}_1 \in \ , {}_2 \in \ \}, \quad (7)$$

$$\vee_{\in} \ = \{ \vee_{\in} \ | \ \in \ \}. \quad (8)$$

RESULTS

This section will cover the fundamental characteristics of JU-algebra and associated findings. The findings of JU-algebra, JU-subalgebra, JU-ideal, JU-weak ideal, and JU-strong ideal with respect to the CSCs associated with ς-level under FRs have been examined. Furthermore, this section will cover the fundamental characteristics of JU-ideals in quantales and associated findings.

Proposition 3.1

Let (, $\mathcal{SC}_\eta($;ς)) be an $\mathcal{SC}_\eta($;ς)-AS of CPFR. Then for all $_1$, $_2$ ∈ , we have

$$\mathcal{SC}_\eta(b_1;ς) \circledast \mathcal{SC}_\eta(b_2;ς) \subseteq \mathcal{SC}_\eta(b_1 \circledast b_2;ς) \quad (9)$$

Example 3.1 Let the following Table 4 define 𝕁𝕌-subalgebra on ⊛ as = {0, , , , , }.

⊛						
	0					
	0	0				
	0	0	0			
	0	0	0	0		
	0	0	0		0	
	0	0	0	0	0	0

Table 4: On the set , a binary operation ⊛ is defined.

For every two components in , define the membership grades of their connection under η on has the following relations:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

η is obviously compatible. The SCs of each element in associated with 0.6 under η for ς=0.6 are:

$$\mathcal{SC}_{\eta}(0; 0.6) = \{0\},$$

$$\mathcal{SC}_{\eta}(\;; 0.6) = \{ \;, \; \},$$

$$\mathcal{SC}_{\eta}(\;; 0.6) = \{ \;, \; \},$$

$$\mathcal{SC}_{\eta}(\;; 0.6) = \{ \; \},$$

$$\mathcal{SC}_{\eta}(\;; 0.6) = \{ \;, \; \} \text{ and}$$

$$\mathcal{SC}_{\eta}(\;; 0.6) = \{ \; \}.$$

Therefore, each element CSCs in that is connected to the 0.6 level under η are:

$$\mathcal{CS}\ _{\eta}(0; 0.6) = \{0\},$$

$$\mathcal{CSC}_{\eta}(\;; 0.6) = \{ \;, \;, \; \},$$

$$\mathcal{CSC}_{\eta}(\;; 0.6) = \{ \;, \;, \; \},$$

$$\mathcal{CS}\ _{\eta}(\;; 0.6) = \{ \; \},$$

$$\mathcal{CSC}_{\eta}(\;; 0.6) = \{ \;, \;, \; \} \text{ and}$$

$$\mathcal{CS}\ _{\eta}(\;; 0.6) = \{ \; \}.$$

Here, it is simple to confirm that $\forall \ _1, \ _2 \in \ ,$

$$\mathcal{SC}_{\eta}(b_1; \varsigma) \circledast \ \mathcal{SC}_{\eta}(b_2; \varsigma) \subseteq \ \mathcal{SC}_{\eta}(b_1 \circledast b_2; \varsigma).$$

Note that equality in general does not apply in this case. Let's look at the example that holds equality is as follows.

Example 3.2 Using the following **Error! Reference source not found.** 5 and let $= \{0, \;, \;, \;, \;, \; \}$ be 𝕁𝕌-subalgebra defined on ⊛.

⊛						
	0					
	0	0				
	0	0	0			
	0	0	0	0		
	0	0	0		0	
	0	0	0	0	0	0

For every two components in , define the membership grades of their connection under η on has the following relations:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Here η is compatible. For ς=0.6, any element in that is connected to 0.6 under η has a SCs that are:

$$\mathcal{SC}_{\eta}(0; 0.6) = \{\ \},$$

$$\mathcal{SC}_{\eta}(\ ; 0.6) = \{\ ,\ ,\ \},$$

$$\mathcal{SC}_{\eta}(\ ; 0.6) = \{\ ,\ ,\ \},$$

$$\mathcal{SC}_{\eta}(\ ; 0.6) = \{\ ,\ ,\ \},$$

$$\mathcal{SC}_{\eta}(\ ; 0.6) = \{\ \} \text{ and}$$

$$\mathcal{SC}_{\eta}(\ ; 0.6) = \{\ \}.$$

Accordingly, each member in CSCs at the 0.6 level under η are:

$$\mathcal{CS}\ _{\eta}(0; 0.6) = \{0,\ \},$$

$$\mathcal{CSC}_{\eta}(\ ; 0.6) = \{\ ,\ ,\ \},$$

$$\mathcal{CS}\ _{\eta}(\ ; 0.6) = \{\ ,\ ,\ \},$$

$$\mathcal{CS}\ _{\eta}(\ ; 0.6) = \{\ ,\ ,\ \},$$

$$\mathcal{CS}\ _{\eta}(\ ; 0.6) = \{0,\ \} \text{ and}$$

$$\mathcal{CS}\ _{\eta}(\ ; 0.6) = \{\ \}.$$

It is simple to confirm here that $\forall\ _1,\ _2 \in\ ,$

$$\mathcal{SC}_{\eta}(b_1; \varsigma) \circledast\ \mathcal{SC}_{\eta}(b_2; \varsigma) =\ \mathcal{SC}_{\eta}(b_1 \circledast b_2; \varsigma).$$

Remark 3.1

The FR matrices used in the examples are constructed mathematical examples rather than arbitrary empirical datasets. They are particularly chosen to fulfill assumptions used in the definitions and theorems of the corresponding classes and to facilitate the construction of SCs, CSCs, lower and upper approximation.

Definition 3.1

Let (, $\mathcal{SC}_{\eta}(\ ; \varsigma)$) be an $\mathcal{SC}_{\eta}(\ ; \varsigma)$-AS of CPFR. If for all $\ _1,\ _2 \in\ ,$

$$\mathcal{SC}_{\eta}(b_1; \varsigma) \circledast\ \mathcal{SC}_{\eta}(b_2; \varsigma) =\ \mathcal{SC}_{\eta}(b_1 \circledast b_2; \varsigma) \tag{10}$$

Then the collection $\mathcal{SC}_{\eta}(\ ; \varsigma)$ is called $\circledast$ −Complete FR (CFR).

Let (, $\mathcal{SC}_{\eta}(\ ; \varsigma)$) be an $\mathcal{SC}_{\eta}(\ ; \varsigma)$-AS of CPFR. If for all $\ _1,\ _2 \in\ ,$

$$\mathcal{SC}_{\eta}(b_1; \varsigma) \vee\ \mathcal{SC}_{\eta}(b_2; \varsigma) =\ \mathcal{SC}_{\eta}(b_1 \vee b_2; \varsigma) \tag{11}$$

Then the collection $\mathcal{CS}\ _{\eta}(\ ; \varsigma)$ is called ∨ −CFR.

If it is both $\circledast$ −Complete and $\vee$ −Complete is called CFR.

Theorem 3.1

Let $(\ ,\ \mathcal{SC}_{\eta}(\ ;\varsigma))$ and $(\ ,\ \mathcal{SC}_{\theta}(\ ;\tau))$ be an $\mathcal{SC}_{\eta}(\ ;\varsigma)$-AS and $\mathcal{SC}_{\theta}(\ ;\tau)$-AS respectively. These characteristics are true if $\phi \neq\ ,\ \subseteq\ $:

1. $\underline{\ }(\ ,\varsigma) \subseteq\ \subseteq \overline{\ }(\ ,\varsigma)$;
2. $\underline{\ }(\phi,\varsigma) = \phi = \overline{\ }(\phi,\varsigma)$;
3. $\overline{\ }(\ \cup\ ,\varsigma) = \overline{\ }(\ ,\varsigma) \cup \overline{\ }(\ ,\varsigma)$;
4. $\underline{\ }(\ \cap\ ,\varsigma) = \underline{\ }(\ ,\varsigma) \cap \underline{\ }(\ ,\varsigma)$;
5. $\subseteq$ implies that $\underline{\ }(\ ,\varsigma) \subseteq \underline{\ }(\ ,\varsigma)$ and $\overline{\ }(\ ,\varsigma) \subseteq \overline{\ }(\ ,\varsigma)$.
6. $\underline{\ }(\ ,\varsigma) \cap \underline{\ }(\ ,\varsigma) \subseteq \underline{\ }(\ \cap\ ,\varsigma)$;
7. $\overline{\ }(\ ,\varsigma) \cup \overline{\ }(\ ,\varsigma) \subseteq \overline{\ }(\ \cup\ ,\varsigma)$;
8. $\eta \subseteq \theta$ and $\varsigma \subseteq \tau$ implies that $\underline{\ }(\ ;\tau) \subseteq \underline{\ }(\ ;\varsigma)$ and $\overline{\ }(\ ;\varsigma) \subseteq \overline{\ }(\ ;\tau)$.

Proof

(1) If $\in \underline{\ }(\ ;\varsigma)$, then $\in CS_{\eta}(\ ;\varsigma) \subseteq\ $. This implies that $\in CS_{\eta}(\ ;\varsigma)$ and $\in\ $. From there we can write $CS_{\eta}(\ ;\varsigma) \cap\ \neq \phi$. This implies that $\in \overline{\ }(\ ;\varsigma)$. Hence, $\subseteq \overline{\ }(\ ,\varsigma)$. Consequently, $\underline{\ }(\ ,\varsigma) \subseteq\ \subseteq \overline{\ }(\ ,\varsigma)$.

(2) It is straightforward.

(3) Let,

$$\in \overline{\ }(\ \cup\ ,\varsigma) \Leftrightarrow CS_{\eta}(\ ;\varsigma) \cap (\ \cup\ ) \neq \phi$$

$$\Leftrightarrow (CS_{\eta}(\ ;\varsigma) \cap\ ) \cup (CS_{\eta}(\ ;\varsigma) \cap\ ) \neq \phi$$

$$\Leftrightarrow CS_{\eta}(\ ;\varsigma) \cap\ \neq\ or\ CS_{\eta}(\ ;\varsigma) \cap\ \neq \phi$$

$$\Leftrightarrow\ \in \overline{\ }(\ ;\varsigma) \text{ or } \in \overline{\ }(\ ;\varsigma)$$

$$\Leftrightarrow\ \in \overline{\ }(\ ;\varsigma) \cup \overline{\ }(\ ;\varsigma).$$

Thus,

$$\overline{\ }(\ \cup\ ,\varsigma) = \overline{\ }(\ ,\varsigma) \cup \overline{\ }(\ ,\varsigma).$$

(4) Let,

$$\in \underline{\ }(\ \cap\ ,\varsigma) \Leftrightarrow CS_{\eta}(\ ;\varsigma) \subseteq (\ \cap\ )$$

$$\Leftrightarrow CS_{\eta}(\ ;\varsigma) \subseteq\ and\ CS_{\eta}(\ ;\varsigma) \subseteq$$

$$\Leftrightarrow\ \in \underline{\ }(\ ;\varsigma) \text{ and } \in \underline{\ }(\ ;\varsigma)$$

$$\Leftrightarrow\ \in \underline{\ }(\ ;\varsigma) \cap \underline{\ }(\ ;\varsigma).$$

Thus,

$$\_(\ \cap\ ,\varsigma) = \_(\ ,\varsigma) \cap \_(\ ,\varsigma).$$

(5) Since $\subseteq$ if and only if $\cap$ $=$, by (3) we have

$$\_(\ ,\varsigma)=\_(\ \cap\ ,\varsigma) = \_(\ ,\varsigma) \cap \_(\ ,\varsigma)$$

This implies that $\_(\ ,\varsigma) \subseteq \_(\ ,\varsigma)$. Additionally, take note that $\subseteq$ if and only if $\cap$ $=$. From (2), we obtain $\_(\ ,\varsigma)=\ (\_\ \cup\ ,\varsigma)=\ (\ ,\varsigma) \cup\ (\ ,\varsigma)$.

This implies that $(\ ,\varsigma)\subseteq\ (\ ,\varsigma)$.

(6) Since $\subseteq$ $\cup$ and $\subseteq$ $\cup$, by (4) we have

$$\_(\ ,\varsigma)=\_(\ \cap\ ,\varsigma) \text{ and } \_(\ ,\varsigma)=\_(\ \cap\ ,\varsigma)$$

This implies that $\_(\ ,\varsigma) \cup \_(\ ,\varsigma)=\_(\ \cap\ ,\varsigma)$.

(7) Since $\cap$ $\subseteq$ and $\cap$ $\subseteq$, by (4) we have

$$\bar{}(\ \cap\ ,\varsigma) \subseteq \bar{}(\ ,\varsigma) \text{ and } \bar{}(\ \cap\ ,\varsigma) \subseteq \bar{}(\ ,\varsigma)$$

which implies that $\bar{}(\ \cap\ ,\varsigma) \subseteq \bar{}(\ ,\varsigma) \cap \bar{}(\ ,\varsigma)$.

(8) Since $\varsigma \subseteq \tau$. If $x \in \_(\ ;\tau)$, then $CS_{\theta}(x;\tau) \subseteq$. But $\varsigma \subseteq \tau$. Therefore, $CS_{\eta}(x;\varsigma) \subseteq CS_{\theta}(x;\tau) \subseteq$.

This implies that $CS_{\eta}(x;\varsigma) \subseteq$. Thus, $x \in \_(\ ;\varsigma)$. Hence, $\_(\ ;\tau)\subseteq \_(\ ;\varsigma)$.

Now, let $x \in \bar{}(\ ;\varsigma)$, then we have $CS_{\eta}(x;\varsigma) \cap\ \neq$.

Then $\exists\ y \in CS_{\eta}(x;\varsigma) \cap$ s. t. $y \in CS_{\eta}(x;\varsigma)$ and $y \in$. Hence, $SC_{\eta}(y;\varsigma) = SC_{\eta}(x;\varsigma)$, Since $\varsigma \subseteq \tau$, so we have $y \in CS_{\theta}(x;\tau)$.

Therefore, we have $y \in CS_{\theta}(x;\tau) \cap$.

Which means that $x \in \bar{}(\ ;\tau)$.

Hence, $\bar{}(\ ;\varsigma) \subseteq \bar{}(\ ;\tau)$.

Theorem 3.2 Let $(\ ,\ SC_{\eta}(\ ;\varsigma))$ be an $SC_{\eta}(\ ;\varsigma)$-AS. If are nonempty subsets of , then

1. $\_(\_(\ ,\varsigma),\varsigma)=\underline{\eta}(\ ,\varsigma)$
2. $\bar{}(\bar{}(\ ,\varsigma),\varsigma)=\overline{\eta}(\ ,\varsigma)$
3. $\bar{}(\_(\ ,\varsigma),\varsigma)=\underline{\eta}(\ ,\varsigma)$
4. $\_(\bar{}(\ ,\varsigma),\varsigma)=\overline{\eta}(\ ,\varsigma)$
5. $\_(\ ,\varsigma)=((\bar{}(\ ^{c},\varsigma))$
6. $\bar{}(\ ,\varsigma)=((\_(\ ^{c},\varsigma))$
7. $\_(CS_{\eta}(\ ;\varsigma),\varsigma)=\ =\_(CS_{\eta}(\ ;\varsigma),\varsigma)$, for all $\in$.

Proof

Proof is simple.

Theorem 3.3

Let (, $\mathcal{SC}_\eta$(; ς)) be an $\mathcal{SC}_\eta$(; ς)-AS. If ϕ ≠ ⊆ , then

1. ¯(, ς)⊛ ¯(, ς)⊆ ¯(⊛ , ς)
2. If η is CFR, then _(, ς) ⊛ _(, ς) ⊆ _(⊛ , ς).

Proof

(1) Any element of ¯(, ς)⊛ ¯(, ς) can be . Consequently, = ⊛ , where ∈ ¯(, ς) and ∈ ¯(, ς). As a result, , exist in such that ∈ CSC_η(, ς) ∩ and ∈ CSC_η(, ς) ∩ . It follows from this that ∈ CSC_η(, ς), ∈ and ∈ CSC_η(, ς), ∈ .

Proposition 3. gives us

⊛ ∈ CSC_η(, ς) ⊛ CSC_η(, ς) ⊆ CSC_η(⊛ , ς)

Furthermore, since we have ⊛ ∈ ⊛ , ⊛ ∈ CSC_η(⊛ , ς). This suggests that ⊛ ∈ CSC_η(⊛ , ς) ∩ ⊛ . This implies that = ⊛ ∈ ¯(⊛ , ς). Thus

¯(, ς)⊛ ¯(, ς)⊆ ¯(⊛ , ς)

(2) Let be any element of _(, ς)⊛ _(, ς). Then = ⊛ with ∈ _(, ς) and ∈ _(, ς). This follows that CSC_η(, ς) ⊆ and CSC_η(, ς) ⊆ . Since η is CFR, so we have

CSC_η(, ς) ⊛ CSC_η(, ς) = CSC_η(⊛ , ς) ⊆ ⊛ .

This implies that = ⊛ ∈ _(⊛ , ς). Thus,

_(, ς)⊛ _(, ς)⊆ _(⊛ , ς).

Theorem 3.4

Let is a 𝕁𝕌-subalgebra of and (, $\mathcal{SC}_\eta$(; ς)) be an $\mathcal{SC}_\eta$(; ς)-AS of ⊛ −CFR, then

1. ¯(, ς) is a 𝕁𝕌-subalgebra of .
2. _(, ς) is a 𝕁𝕌-subalgebra of .

Proof

(1) Assume that , are in ¯(, ς). Then CSC_η(, ς) ∩ ≠ and CSC_η(ŋ, ς) ∩ ≠ , and so there exist , ∈ such that ∈ CSC_η(, ς) and ∈ CSC_η(ŋ, ς). According to Proposition 3., we have

⊛ ∈ CSC_η(, ς) ⊛ CSC_η(, ς) ⊆ CSC_η(⊛ ŋ, ς).

It follows from this that ⊛ ∈ CSC_η(⊛ ŋ, ς). Given that represent a 𝕁𝕌-subalgebra, ⊛ ∈ . Accordingly, ⊛ ∈ CSC_η(⊛ ŋ, ς) ∩ , that is CSC_η(⊛ ŋ, ς) ∩ ≠ . In this case, ⊛ ∈ ¯(, ς). Hence, ¯(, ς) is a 𝕁𝕌-subalgebra of .

(2) Assume that , ∈ _(, ς). Then CSC_η(, ς) ⊆ and CSC_η(ŋ, ς) ⊆ , respectively. Given that is a 𝕁𝕌-subalgebra of and η is ⊛ −CFR, so

$$\mathrm{CSC}_{\eta}(\ ,\varsigma) \circledast \mathrm{CSC}_{\eta}(\mathfrak{y},\varsigma) = \mathrm{CSC}_{\eta}(\ \circledast \mathfrak{y},\varsigma) \subseteq$$

Hence $\circledast \ \in \_(\ ,\varsigma)$. Thus, $\_(\ ,\varsigma)$ is 𝕁𝕌-subalgebra of .

Theorem 3.5

If is a 𝕁𝕌-ideal of and (, $\mathcal{SC}_{\eta}(\ ;\varsigma)$) be an $\mathcal{SC}_{\eta}(\ ;\varsigma)$-AS of $\circledast$ −CFR, then

1. $^{-}(\ ,\varsigma)$ is a 𝕁𝕌-ideal of .
2. $\_(\ ,\varsigma)$ is a 𝕁𝕌-ideal of .

Proof

(1) Consider is a 𝕁𝕌-ideal of . Indeed, $0 \in {}^{-}(\ ,\varsigma)$. For $\ ,\mathfrak{y},\ \in$, let $\ \in {}^{-}(\ ,\varsigma)$ and $\circledast$ ($\circledast$) $\in {}^{-}(\ ,\varsigma)$. After that,

$$\mathrm{CSC}_{\eta}(\mathfrak{y},\ ) \cap \ \neq \ \ and\ \mathrm{CSC}_{\eta}(\ \circledast(\ \circledast\ ),\ ) \cap \ \neq\ ,$$

Thus, , $\in$ such that $\in \mathrm{CSC}_{\eta}(\mathfrak{y},\ )$ and $\in \mathrm{CSC}_{\eta}(\ \circledast(\ \circledast z),\ )$ respectively. Since $\mathrm{SC}_{\eta}(\ ,\ ) = \mathrm{SC}_{\eta}(\mathfrak{y},\ )$ and $\mathrm{SC}_{\eta}(\ ,\ ) = \mathrm{SC}_{\eta}(\ \circledast(\ \circledast\ ),\ )$, it follows that $\circledast \ \in \mathrm{CSC}_{\eta}(\mathfrak{y},\ ) \subseteq$ and $\circledast(\ \circledast\ )\circledast\ \in \mathrm{CSC}_{\eta}(\ \circledast(\ \circledast\ ),\ ) \subseteq$. Given that is a 𝕁𝕌-ideal and , $\in$,

$$\in \ \text{and}\ \circledast(\ \circledast\ ) \in\ ,$$

According to the definition of 𝕁𝕌-ideal, $\circledast \ \in$. $\circledast \ \in \mathrm{CSC}_{\eta}(\ \circledast\ ,\ )$ should be noted. Accordingly, $\circledast \ \in \mathrm{CSC}_{\eta}(\ \circledast\ ,\ ) \cap$ or $\mathrm{CSC}_{\eta}(\ \circledast\ ,\ ) \cap \ \neq$. Thus, $\circledast \ \in {}^{-}(\ ,\varsigma)$.

Hence, $^{-}(\ ,\varsigma)$ is a 𝕁𝕌-ideal of .

(2) Let be 's 𝕁𝕌-ideal. Needless to say, $0 \in \_(\ ,\varsigma)$. Let $\ ,\mathfrak{y},\ \in$ be such that $\in \_(\ ,\varsigma)$ and $\circledast$ ($\circledast$) $\in \_(\ ,\varsigma)$. Next,

$$\mathrm{CSC}_{\eta}(\mathfrak{y},\ ) \subseteq \ \ and\ \mathrm{CSC}_{\eta}(\ \circledast(\ \circledast\ ),\ ) \subseteq\ ,$$

Let's say that $w \in \mathrm{CSC}_{\eta}(\ \circledast\ ,\ ) = \mathrm{CSC}_{\eta}(\ ,\ ) \circledast \mathrm{CSC}_{\eta}(\ ,\ )$. Then, for any $\in \mathrm{CSC}_{\eta}(\ ,\ )$ and $\in \mathrm{CSC}_{\eta}(\ ,\ )$, we have $w \in \mathrm{CSC}_{\eta}(\ ,\ ) \circledast \mathrm{CSC}_{\eta}(\ ,\ )$. $\mathrm{SC}_{\eta}(\ ,\ ) = \mathrm{SC}_{\eta}(\ ,\ )$ and $\mathrm{SC}_{\eta}(\ ,\ ) = \mathrm{SC}_{\eta}(\ ,\ )$ are our two variables. We obtain $\mathrm{SC}_{\eta}(\ ,\ ) = \mathrm{SC}_{\eta}(\mathfrak{y},\ )$ for $\in \mathrm{CSC}_{\eta}(\mathfrak{y},\ )$. Given that η is $\circledast$ −CFR,

$$\mathrm{SC}_{\eta}(\ \circledast(\ \circledast\ ),\ ) = \mathrm{SC}_{\eta}(\ \circledast(\ \circledast\ ),\ )$$

Our result is $\circledast(\ \circledast\ ) \in \mathrm{CSC}_{\eta}(\ \circledast(\ \circledast\ ),\ ) \subseteq$. Given that is a 𝕁𝕌-ideal of , $w = \ \circledast \ \in$ according to the definition of a 𝕁𝕌-ideal.

Therefore, $\mathrm{CSC}_{\eta}(\ \circledast\ ,\ ) \subseteq$. Because of this, $\circledast \ \in \_(\ ,\varsigma)$, and $\_(\ ,\varsigma)$ is a 𝕁𝕌-ideal of .

Theorem 3.6:

If is a weak 𝕁𝕌-ideal of and (, $\mathcal{SC}_{\eta}(\ ;\varsigma)$) be an $\mathcal{SC}_{\eta}(\ ;\varsigma)$-AS of CFR, then

1. $(\ ,\varsigma)$ is a weak 𝕁𝕌- ideal of .
2. $\_(\ ,\varsigma)$ is a weak 𝕁𝕌- ideal of .

Proof:

(1) Let represent 's weak 𝕁𝕌-ideal. Indeed, $0 \in \overline{\ }(\ ,\varsigma)$. Assume , belong to such that $\in$ $\overline{\ }(\ ,\varsigma)$ and $\circledast$ $\in \overline{\ }(\ ,\varsigma)$. After that,

$$\mathrm{CSC}_\eta(\mathfrak{y}, \ ) \cap \ \neq \quad and\ \mathrm{CSC}_\eta(\ \circledast \ , \ ) \cap \ \neq \ ,$$

Consequently, , $\in$ such that $\in \mathrm{CSC}_\eta(\mathfrak{y}, \ )$ and $\in \mathrm{CSC}_\eta(\ \circledast x, \ )$. Thus, according to the equations $\mathrm{SC}_\eta(\ , \ ) = \mathrm{SC}_\eta(\mathfrak{y}, \ )$ and $\mathrm{SC}_\eta(\ , \ ) = \mathrm{SC}_\eta(\ \circledast \ , \ )$, $\circledast$ $\in \mathrm{CSC}_\eta(\ , \ ) \subseteq$ and ($\circledast$) $\circledast$ $\in \mathrm{CSC}_\eta(\ \circledast \ , \ ) \subseteq$. Since $\in$ and $\circledast$ $\in$, and since , $\in$ and is a weak 𝕁𝕌-ideal, the weak 𝕁𝕌-ideal definition indicates that $\in$.

Keep in mind that $\in \mathrm{CSC}_\eta(\ , \ )$. This suggests that $\in \mathrm{CSC}_\eta(\ , \ ) \cap$ or $\mathrm{CSC}_\eta(\ , \ ) \cap \ \neq$. Therefore, $\in \overline{\ }(\ ,\varsigma)$. Hence, $\overline{\ }(\ ,\varsigma)$ is a weak 𝕁𝕌-ideal of .

(2) Let be 's weak 𝕁𝕌-ideal. Of course, $0 \in \underline{\ }(\ ,\varsigma)$.

If , $\in$, then $\in \underline{\ }(\ ,\varsigma)$ and $\circledast$ $\in \underline{\ }(\ ,\varsigma)$. Then,

$$\mathrm{CSC}_\eta(\ , \ ) \subseteq \quad and\ \mathrm{CSC}_\eta(\mathfrak{y}, \ ) \circledast \mathrm{CSC}_\eta(\ , \ ) = \mathrm{CSC}_\eta(\ \circledast \ , \ ) \subseteq \ ,$$

Assuming that is a member of $\mathrm{CSC}_\eta(\ , \ )$, we obtain $\mathrm{SC}_\eta(\ , \ ) = \mathrm{SC}_\eta(\ , \ )$ for some $\in$ $\mathrm{CSC}_\eta(\ , \ )$. Taking $\in \mathrm{CSC}_\eta(\mathfrak{y}, \ )$, we obtain $\mathrm{SC}_\eta(\ , \ ) = \mathrm{SC}_\eta(\ , \ )$. Since η is $\circledast$ −CFR,

$$\mathrm{SC}_\eta(\ \circledast \ , \ ) = \mathrm{SC}_\eta(\ \circledast \ , \ )$$

Therefore, $\circledast$ $\in \mathrm{CSC}_\eta(\ \circledast \ , \ ) \subseteq$. According to the definition of a weak 𝕁𝕌-ideal and $=$ $\in$. Since is a weak 𝕁𝕌-ideal of . Thus, $\mathrm{CSC}_\eta(\ , \ ) \subseteq$. Because is a member of $\underline{\ }(\ ,\varsigma)$. Hence, $\underline{\ }(\ ,\varsigma)$ is a weak 𝕁𝕌-ideal of .

Theorem 3.7

If is a strong 𝕁𝕌-ideal of and (, $\mathcal{SC}_\eta(\ ;\varsigma)$) be an $\mathcal{SC}_\eta(\ ;\varsigma)$-AS of CFR, then

1. $\overline{\ }(\ ,\varsigma)$ referred as a strong 𝕁𝕌- ideal of .
2. $\underline{\ }(\ ,\varsigma)$ referred as a strong 𝕁𝕌- ideal of .

Proof. (1) Assume that , $\in$

$$\circledast \ \cap \overline{\eta}(\ ,\varsigma) \text{ and } \in \overline{\eta}(\ ,\varsigma).$$

Then $\mathrm{CSC}_\eta(\mathfrak{y}, \ ) \cap \ \neq$ *and* $\mathrm{CSC}_\eta(\ \circledast \ , \ ) \cap \ \neq$, and $\exists$ $\in$ such that $=$ $\circledast$ and $\in$ $\overline{\eta}(\ ,\varsigma)$. This implies that, $\mathrm{CSC}_\eta(\ , \ ) \cap \ \neq$ and $\exists$, $\in$ s.t. $\in \mathrm{CSC}_\eta(\ , \ ) \cap$ and $\in$ $\mathrm{CSC}_\eta(y, \ ) \cap$.

Hence $\mathrm{SC}_\eta(\ , \ ) = \mathrm{SC}_\eta(\ , \ )$ and $\mathrm{SC}_\eta(\ , \ ) = \mathrm{SC}_\eta(\ , \ )$, where , $\in$. Consequently, this suggests that $\circledast$ $\in \mathrm{CSC}_\eta(\ , \ ) \subseteq$ and $\circledast$ $\in \mathrm{CSC}_\eta(y, \ ) \subseteq$. is a strong 𝕁𝕌-ideal, and , , , and are all in S.

We have thus demonstrated that $\circledast$ $\cap \mathrm{CSC}_\eta(\mathfrak{y}, \ ) \neq \phi$ and $\in \mathrm{CSC}_\eta(\mathfrak{y}, \ )$. Since is a strong 𝕁𝕌-ideal, $\in$, and so $\mathrm{CSC}_\eta(\ , \ ) \cap \ \neq$.

That is, $\overline{\eta}(\ ,\varsigma)$ is a strong 𝕁𝕌-ideal of .

(2) If , $\in$, then $\in \underline{\ }(\ ,\varsigma)$ and $\circledast$ $\cap \underline{\ }(\ ,\varsigma)$. Assume that and are both belonging to

CSC_η(,) and CSC_η(ŋ,) respectively. Our SC_η(,) = SC_η(,) and SC_η(,) = SC_η(ŋ,) are as follows.

Because η is CFR, SC_η(⊛ ,) = SC_η(⊛ ,). A in such that ∈ ⊛ and ∈ _(,ς) exists since ⊛ ∩ _(,ς). The statement ∈ SC_η(⊛ ,) = SC_η(⊛ ,) now suggests that there is a ∈ SC_η(⊛ ,) such that SC_η(,) = SC_η(,), and therefore CSC_η(,) = SC_η(,) ∈ . As a result, ∈ and (⊛)∩≠ ϕ.

Alternatively, we have ∈ CSC_η(ŋ,) ⊆ . Since ∈ , which suggests that CSC_η(,) ⊆ , is a strong -ideal of .

This indicates that _(,ς) is a strong 𝕁𝕌-ideal of , since belongs to _(,ς).

Remark 3.2

A quantale is a more richly-structured algebraic setting in which the underlying structure is an ordered complete-lattice, along with an associative multiplication which distributes over all joins. This additional structure will provide a natural context for the study of whether the proposed rough approximations and JU-ideal structures are compatible with stronger order-algebraic properties.

Theorem 3.8

Assume that $\dot{\mathbb{Q}}$ is quantale and ϕ ≠ ⊆ $\dot{\mathbb{Q}}$. If is a 𝕁𝕌-ideal of $\dot{\mathbb{Q}}$ and ($\dot{\mathbb{Q}}$, $\mathcal{SC}_\eta$($\dot{\mathbb{Q}}$;ς)) be an $\mathcal{SC}_\eta$($\dot{\mathbb{Q}}$;ς)-AS of CFR, then

‾(,ς) referred as 𝕁𝕌-ideal of $\dot{\mathbb{Q}}$.

_(,ς) referred as 𝕁𝕌-ideal of $\dot{\mathbb{Q}}$.

Proof

(1) Consider is a 𝕁𝕌-ideal of $\dot{\mathbb{Q}}$. Indeed, 0 ∈ ‾(,ς). For ,ŋ, ∈ $\dot{\mathbb{Q}}$, let ∈ ‾(,ς) and ⊛ (⊛) ∈ ‾(,ς). After that, CSC_η(ŋ,) ∩ ≠ *and* CSC_η(⊛ (⊛),) ∩ ≠ ,

Thus, , ∈ such that ∈ CSC_η(ŋ,) and ∈ CSC_η(⊛ (⊛ z),) respectively.

Since SC_η(,) = SC_η(,) and SC_η(,) = SC_η(⊛ (⊛),), it follows that ⊛ ∈ CSC_η(ŋ,) ⊆ and ⊛ (⊛) ⊛ ∈ CSC_η(⊛ (⊛),) ⊆ .

Given that is a 𝕁𝕌-ideal and , ∈ , ∈ and ⊛ (⊛) ∈ , According to the definition of 𝕁𝕌-ideal, ⊛ ∈ . ⊛ ∈ CSC_η(⊛ ,) should be noted.

Accordingly, ⊛ ∈ CSC_η(⊛ ,) ∩ or CSC_η(⊛ ,) ∩ ≠ . Thus, ⊛ ∈ ‾(,ς). Similarly, we can prove for ∨. Hence ‾(,ς) is 𝕁𝕌-ideal of $\dot{\mathbb{Q}}$.

(2) Let be $\dot{\mathbb{Q}}$'s 𝕁𝕌- ideal. Needless to say, 0 ∈ _(,ς). Let ,ŋ, ∈ $\dot{\mathbb{Q}}$ be such that ∈ _(,ς) and ⊛ (⊛) ∈ _(,ς). Next,

CSC_η(ŋ,) ⊆ *and* CSC_η(⊛ (⊛),) ⊆ ,

Suppose that w ∈ CSC_η(⊛ ,) = CSC_η(,) ⊛ CSC_η(,). Then for any ∈ CSC_η(,) and ∈ CSC_η(,), we have w ∈ CSC_η(,) ⊛ CSC_η(,). SC_η(,) = SC_η(,) and SC_η(,) = SC_η(,) are our two variables. We obtain SC_η(,) = SC_η(ŋ,) for ∈ CSC_η(ŋ,).

Given that η is CFR, so

$$SC_\eta(\ \circledast(\ \circledast\ ),\ ) = SC_\eta(\ \circledast(\ \circledast\ ),\ )$$

Our result is $\circledast(\ \circledast\ ) \in CSC_\eta(\ \circledast(\ \circledast\ ),\ ) \subseteq$. Given that is a 𝕁𝕌-ideal of $\dot{\mathbb{Q}}$, w = $\circledast$ $\in$ according to the definition of a 𝕁𝕌-ideal.

Therefore, $CSC_\eta(\ \circledast\ ,\ ) \subseteq$. Because of this, $\circledast$ $\in \underline{\ }(\ ,\varsigma)$.

Similarly, we can prove for ∨. Hence, $\underline{\ }(\ ,\varsigma)$ is a 𝕁𝕌-ideal of $\dot{\mathbb{Q}}$.

Theorem 3.9

If is a weak 𝕁𝕌-ideal of $\dot{\mathbb{Q}}$ and ($\dot{\mathbb{Q}}$, $\mathcal{SC}_\eta(\dot{\mathbb{Q}}\ ;\varsigma)$) be an $\mathcal{SC}_\eta(\dot{\mathbb{Q}}\ ;\varsigma)$-AS of CFR, then

1. $\overline{\ }(\ ,\varsigma)$ referred as a weak 𝕁𝕌-ideal of $\dot{\mathbb{Q}}$.
2. $\underline{\ }(\ ,\varsigma)$ referred a weak 𝕁𝕌-ideal of $\dot{\mathbb{Q}}$.

Proof

(1) Let represent $\dot{\mathbb{Q}}$'s weak 𝕁𝕌-ideal. Indeed, $0 \in \overline{\ }(\ ,\varsigma)$. Assume , belong to $\dot{\mathbb{Q}}$ such that $\in \overline{\ }(\ ,\varsigma)$ and $\circledast$ $\in \overline{\ }(\ ,\varsigma)$. This implies that,

$$CSC_\eta(\mathfrak{y},\ ) \cap \quad \neq \quad \mathit{and}\ CSC_\eta(\ \circledast\ ,\ ) \cap \quad \neq \ ,$$

Consequently, , $\in$ such that $\in CSC_\eta(\mathfrak{y},\ )$ and $\in CSC_\eta(\ \circledast x,\ )$. Thus, according to the equations $SC_\eta(\ ,\ ) = SC_\eta(\mathfrak{y},\ )$ and $SC_\eta(\ ,\ ) = SC_\eta(\ \circledast\ ,\ )$, $\circledast$ $\in CSC_\eta(\ ,\ ) \subseteq$ and ($\circledast$) $\circledast$ $\in CSC_\eta(\ \circledast\ ,\ ) \subseteq$.

Since $\in$ and $\circledast$ $\in$, and , $\in$ and is a weak 𝕁𝕌-ideal, The weak 𝕁𝕌-ideal definition indicates that $\in$. We know that $\in CSC_\eta(\ ,\ )$. This suggests that $\in CSC_\eta(\ ,\ ) \cap$ or $CSC_\eta(\ ,\ ) \cap \quad \neq$.

Therefore, $\in \overline{\ }(\ ,\varsigma)$. Similarly, we can prove for ∨. Hence, $\overline{\ }(\ ,\varsigma)$ is a weak 𝕁𝕌-ideal of $\dot{\mathbb{Q}}$.

(2) Let be $\dot{\mathbb{Q}}$'s weak 𝕁𝕌-ideal. Of course, $0 \in \underline{\ }(\ ,\varsigma)$. If , $\in \dot{\mathbb{Q}}$, then $\in \underline{\ }(\ ,\varsigma)$ and $\circledast$ $\in \underline{\ }(\ ,\varsigma)$. Then

$$CSC_\eta(\ ,\ ) \subseteq \quad \mathit{and}\ CSC_\eta(\mathfrak{y},\ ) \circledast CSC_\eta(\ ,\ ) = CSC_\eta(\ \circledast\ ,\ ) \subseteq\ ,$$

Assuming that is a member of $CSC_\eta(\ ,\ )$, we obtain $SC_\eta(\ ,\ ) = SC_\eta(\ ,\ )$ for some $\in CSC_\eta(\ ,\ )$. Taking $\in CSC_\eta(\mathfrak{y},\ )$, we obtain $SC_\eta(\ ,\ ) = SC_\eta(\ ,\ )$. Since η is CFR, $SC_\eta(\ \circledast\ ,\ ) = SC_\eta(\ \circledast\ ,\ )$.

Therefore, $\circledast$ $\in CSC_\eta(\ \circledast\ ,\ ) \subseteq$. According to the definition of a weak 𝕁𝕌-ideal, $=$ $\in$. Since is a weak 𝕁𝕌-ideal of $\dot{\mathbb{Q}}$. Thus, $CSC_\eta(\ ,\ ) \subseteq$. Because is a member of $\underline{\ }(\ ,\varsigma)$. Similarly, we can prove for ∨. Hence, $\underline{\ }(\ ,\varsigma)$ is a weak 𝕁𝕌-ideal of $\dot{\mathbb{Q}}$.

Theorem 3.10

If is a strong 𝕁𝕌- ideal of $\dot{\mathbb{Q}}$ and ($\dot{\mathbb{Q}}$, $\mathcal{SC}_\eta(\dot{\mathbb{Q}}\ ;\varsigma)$) be an $\mathcal{SC}_\eta(\dot{\mathbb{Q}}\ ;\varsigma)$-AS of CFR, then

1. $\overline{\ }(\ ,\varsigma)$ referred as a strong 𝕁𝕌- ideal of $\dot{\mathbb{Q}}$.
2. $\underline{\ }(\ ,\varsigma)$ referred as a strong 𝕁𝕌- ideal of $\dot{\mathbb{Q}}$.

Proof

(1) Assume that , ∈ $\dot{\mathbb{Q}}$, (⊛) ∩ $\overline{\eta}$(, ς) and ∈ $\overline{\eta}$(, ς).

Then CSC_{η}(ŋ,) ∩ ≠ *and* CSC_{η}(⊛ ,) ∩ ≠ , and ∃ ∈ $\dot{\mathbb{Q}}$ such that = ⊛ and ∈ $\overline{\eta}$(, ς).

Hence, CSC_{η}(,) ∩ ≠ and ∃ , ∈ s.t. ∈ CSC_{η}(,) ∩ and ∈ CSC_{η}(y,) ∩ .

Hence SC_{η}(,) = SC_{η}(,) and SC_{η}(,) = SC_{η}(,),

where , ∈ .

Consequently, this suggests that ⊛ ∈ CSC_{η}(,) ⊆ and ⊛ ∈ CSC_{η}(y,) ⊆ . is a strong 𝕁𝕌-ideal, and , , , and are all in .

We have thus demonstrated that ⊛ ∩ CSC_{η}(ŋ,) ≠ ϕ and ∈ CSC_{η}(ŋ,).

Since is a strong 𝕁𝕌-ideal, ∈ , and so CSC_{η}(,) ∩ ≠ and ∈ ¯(, ς).

Similarly, we can prove for ∨. Hence, $\overline{\eta}$(, ς) is a strong 𝕁𝕌-ideal of $\dot{\mathbb{Q}}$.

(2) If , ∈ $\dot{\mathbb{Q}}$, then ∈ _(, ς) and ⊛ ∩ _(, ς).

Assume that and are both belonging to CSC_{η}(,) and CSC_{η}(ŋ,) respectively. Our SC_{η}(,) = SC_{η}(,) and SC_{η}(,) = SC_{η}(,) are as follows.

Because η is CFR, SC_{η}(⊛ ,) = SC_{η}(⊛ ,). A in $\dot{\mathbb{Q}}$ such that ∈ ⊛ and ∈ _(, ς) exists, since ⊛ ∩ _(, ς).

The statement " ∈ SC_{η}(⊛ ,) = SC_{η}(⊛ ,)" now suggests that there is a ∈ SC_{η}(⊛ ,) such that SC_{η}(,) = SC_{η}(,), and therefore CSC_{η}(,) = SC_{η}(,) ∈ . As a result, ∈ .

Alternatively, we have ∈ CSC_{η}(ŋ,) ⊆ .

Since ∈ , which suggests that CSC_{η}(,) ⊆ , since is a strong 𝕁𝕌-ideal of $\dot{\mathbb{Q}}$.

Hence, belongs to _(, ς). Similarly, we can prove for ∨.

Hence, _(, ς) is a strong 𝕁𝕌-ideal of $\dot{\mathbb{Q}}$.

COMPARATIVE ANALYSIS AND DISCUSSION

This section is a comparative study of the suggested CSC-FR based approximation framework and discusses the conceptual advantages of this framework over existing frameworks. In particular, we point out the limitations of the quantale rough approximation schemes of congruence-based and of soft-relation-based, the research gap, and the motivation for the study of rough approximations of quantale substructures through CSCs under FRs.

The proposed one is a contribution to the theory of quantales and other suggestions of ideals, where LA and UA are introduced to the major substructures, such as $\mathbb{JU}$-subalgebras, $\mathbb{JU}$-ideals, weak $\mathbb{JU}$-ideals and strong $\mathbb{JU}$-ideals. It is based on core neighborhood structures induced by FRs to provide a more flexible approximation mechanism in quantales, rather than using equivalence or congruence classes. This can be very helpful in situations where the granularity of the partition is too fine, and the core relational relationships require a description of their behavior. The study of $\mathbb{JU}$-subalgebras, $\mathbb{JU}$-ideals, weak $\mathbb{JU}$-ideals, and strong $\mathbb{JU}$-ideals can be helpful in the development of new theoretical tools for representing uncertainty and relational dependence in algebraic systems in a broader way.

Specific comparison of our proposed framework with some of the existing methodologies is as follows:

(1) Generalized soft-relation based approximations of fuzzy substructures in quantales were extended to semigroups and quantales by Qurashi **Error! Reference source not found.**. However, rough approximations of fuzzy $\mathbb{JU}$-substructures using CSCs induced by FRs have not been systematically investigated. In contrast, in this article we explored a few characterizations of quantale substructures induced by the FRs, resulting in an innovative approximation framework for quantales.

(2) Finding suitable equivalence or congruence relations is not typically easy in algebraic structures, and may be limiting for coarse approximation. The existing quantale-roughness paradigms, accordingly, require extra structures specifications **Error! Reference source not found.**. Contrary to this, the presented approach does not need an equivalence or a congruence relation: the theory is placed in a more flexible environment because it uses CSCs that are extracted from FRs.

(3) The rough approximation notion has been applied in some previous works on rough approximations in quantales and semigroups using soft relations, in which aftersets and foresets are used **Error! Reference source not found.**,33]. The approach in the present article consists in the use of approximations of quantale substructures based on CSCs created from FRs (or, if necessary, on associated compatibility conditions). This is a different approach from the partition classes or parameterized soft families which are typically used.

(4) Furthermore, Qurashi and Kanwal and Shabir studied approximation methods for fuzzy substructures of semigroups and quantales, respectively, using soft relations (usually aftersets and foresets) [30,34]. The proposed work leads to the development of CSCs from a complete/FR framework (refer to our setting), and uses these CSCs to describe lower and upper approximations of fuzzy subsets and $\mathbb{JU}$-Ideal-type substructures respectively. This CSC-driven methodology is, to our knowledge, an original approach in the context of the quantale environment and is thus a fresh perspective in the comparative context.

(5) A comparison of the conceptual characteristics of the three approximation frameworks (congruence-based, soft relation-based, and the suggested CSC-based model) is recapitulated in Table 6. It is worth noticing that the comparison is performed for a fixed ς-cut level $\varsigma \in [0,1]$. In the proposed framework, we have $SC_{\eta}(\mathfrak{b};\varsigma){:}=\{\ \in \mathcal{X}: \eta(\mathfrak{b},\mathfrak{a}) \geq \varsigma\}$ and $CSC_{\eta}(\ _1;\varsigma){:}=\{\ _2 \in \mathcal{Y}: SC_{\eta}(\ _1;\varsigma) = SC_{\eta}(\ _2;\varsigma)\}$, where η is FR from to .

Feature	Congruence-based	Soft relation-based	OSC-FR-based (Proposed)
Underlying relation	Equivalence or congruence	Soft relation (aftersets/foresets)	FR η: × ⟶[0,1]
Classes type	Partition class [ ]	Parameterized families (e.g., aftersets (( ))	Core families $CS_{\eta}(\ ;\varsigma)$ (generally not a partition)
LA criterion	$\in \underline{\eta}(\ ;\ ) \Leftrightarrow [\ ] \subseteq$	$\in \underline{\eta}(\ ;\ ) \Leftrightarrow (\ ) \subseteq$	$\in \underline{\eta}(\ ;\ ) \Leftrightarrow SC_{\eta}(\ ;\varsigma) \subseteq$
UA criterion	$\in \overline{\eta}(\ ;\ ) \Leftrightarrow [\ ] \cap \neq \emptyset$	$\in \overline{\eta}(\ ;\ ) \Leftrightarrow (\ ) \cap \neq \emptyset$	$\in \overline{\eta}(\ ;\ ) \Leftrightarrow SC_{\eta}(\ ;\varsigma) \cap \neq \emptyset$
Requirements to describe approximations	Existence of a congruence; partition-based structure	Existence of soft relation and parameters; no congruence needed	No congruence required
Handles Core neighbourhoods	No (partition-based)	Limited / parameter-driven	Yes (inherent core via CSCs)
Captures indirect membership	No	No	Yes (membership via core of successor neighborhoods)
Construction difficulty (conceptual)	Often restrictive (finding congruences can be nontrivial)	Moderate (depends on choice of parameters/soft sets)	Flexible (depends mainly on η and ς)

Table 6: Approximation Analysis of Various Models by Comparison

CONCLUSION AND FUTURE RESEARCH DIRECTIONS

In this study, we have formulated a rough approximation system for the 𝕁𝕌-algebraic structures based on CSCs and FR. The primary goal was to create a systematic relationship between the fuzzy relational uncertainty and the algebraic roughness using some information granules such as SCs and CSCs in the construction of lower and upper approximations. In contrast with classical rough-set models using mainly crisp equivalence relations, the proposed models are based on graded relationships, realized by FRs, and the interaction between the approximations and the algebraic structure of 𝕁𝕌-algebras is explored.

The study first established the foundation concepts to build the rough approximation based on CSC. The paper focused on the lower and upper approximations related to 𝕁𝕌-subalgebras and various classes of 𝕁𝕌-ideals, based on these ideas. Specifically, the conditions were determined under which the proposed approximations are 𝕁𝕌-subalgebra, 𝕁𝕌-ideal, weak 𝕁𝕌-ideal and strong 𝕁𝕌-ideal.

The results presented here show that the approximation mechanism proposed is not just a set theoretic construct but one that can interact with the algebraic operations and ideal structures of 𝕁𝕌-algebras. Another contribution of the study is the extension of the proposed framework to quantales. In the quantale setting, the binary operation is coupled with a full-lattice structure, giving the user a more complex algebraic setting to work in. In this context, rough 𝕁𝕌-ideals, rough weak 𝕁𝕌-ideals, and rough strong 𝕁𝕌-ideals are studied and the corresponding structural results are obtained. Therefore, the proposed framework is not limited to the original 𝕁𝕌-algebraic framework, and can be used as a foundation for investigating roughness of CSC in a wider class of ordered algebraic structures.

FUTURE RESEARCH DIRECTIONS

We can design computational processes to generate automatically successor and core successor classes and lower and upper approximations for finite information systems. These algorithms would allow for experimental testing of the results and would allow for systematic comparison with other rough-set approaches.

Future research may be directed towards adapting or designing fuzzy-relation thresholds instead of making them a priori. It can enhance the strength of the approximation mechanism in the case of large or noisy information systems.

The algebraic structure of the suggested rough structures may be further explored by the help of the homomorphisms, congruence relations, quotient structures, closure systems and lattice-theoretic representations.

In conclusion, this study provides a theoretical basis for the fusion of FR induced CSCs and rough JU-algebraic structures and quantales. The results illustrate the possibilities of converting the relational uncertainty into algebraically significant rough structures, which have retained the crucial subalgebraic and ideal features. The proposed framework thus serves as a foundation for further investigations in the area of rough set theory, fuzzy relations, algebraic logic and quantale theory.

AUTHOR CONTRIBUTIONS

M.Y participated in the conceptualization, investigation, validation, visualization, data creation, review, administration, writing the original draft, and editing of the manuscript. C. K participated in the administration, validation, visualization, review and editing of the manuscript. All authors read and approved the final manuscript.

AVAILABILITY OF DATA AND MATERIALS

All data that support the findings of this study are included in the article.

CONFLICTS OF INTEREST

The authors declare no conflict of interest

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