

The Porous - Fisher - KPP Equation: Degenerate Diffusion, Sharp Traveling Fronts, and Spreading Dynamics - A Review

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Received: August 29, 2026; Manuscript No: JMPM-26-2277; Editor Assigned: September 01, 2026; PreQc No: JMPM-26-2277 (PQ); Reviewed: October 08, 2026; Revised: October 12, 2026; Accepted: October 20, 2026; Published: October 30, 2026

Citation: Huang W (2026). The Porous - Fisher - KPP Equation: Degenerate Diffusion, Sharp Traveling Fronts, and Spreading Dynamics - A Review. J. Math. Phys. Mech. Vol.1 Iss.1, October (2026), pp:1-2.

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ABSTRACT

The Porous-Fisher-Kolmogorov-Petrovsky-Piscounov (Porous-Fisher-KPP) equation is a nonlinear degenerate parabolic equation that merges the porous medium equation with the classical Fisher-KPP reaction-diffusion model. It has emerged as a powerful mathematical framework for describing biological invasions, tissue growth, and population dynamics where diffusion depends nonlinearly on the local density. This review provides a comprehensive survey of the mathematical theory surrounding the Porous-Fisher-KPP equation. We begin by tracing the historical origins of the model from the independent contributions of Fisher, Kolmogorov, Petrovskii, Piskunov, and the Barenblatt-Aronson theory of the porous medium equation. We then examine traveling wave solutions, finite speed of propagation, free boundary formulations, and the spreading-vanishing dichotomy. Applications to ecology, cell biology, and tissue engineering are highlighted. Finally, we discuss numerical methods, recent advances in degenerate reaction-diffusion systems, and open problems that remain at the frontier of this active field. The reviewed models, traveling-wave properties, methods, and applications are also compared systematically and summarized in five tables.

Keywords: Porous-Fisher-KPP Equation; Traveling Waves; Nonlinear Diffusion; Population Dynamics; Free Boundary; Spreading Speed; Degenerate Parabolic Equations; Reaction-Diffusion Systems

INTRODUCTION

The interplay between diffusion and reaction lies at the heart of mathematical biology. Among the simplest and most influential models capturing this interplay is the Fisher-KPP equation [1,2], which combines linear diffusion with logistic growth. Its ubiquity across disciplines ~ from genetics to ecology ~ is well documented [3,4]. However, the assumption of constant diffusivity is often unrealistic. In biological tissues, cell migration is influenced by local crowding effects: when the cell density is low, diffusion is vigorous, but when the population is crowded, movement slows. The porous medium equation [5-8] captures exactly this density-dependent diffusion, and its merger with the Fisher-KPP model gives rise to the Porous-Fisher-KPP equation. The resulting equation has been shown to exhibit invasion thresholds and spreading dynamics governed by both the diffusion structure and the reaction kinetics, as established by Diekmann and Kaper, and further developed in the ecological context by Hamilton and Hutson [9-10].

The Porous-Fisher-KPP equation belongs to a broader class of degenerate reaction-diffusion equations, where the diffusion coefficient vanishes at zero density. This degeneracy has profound mathematical consequences. Unlike the classical heat equation, solutions can develop sharp interfaces ~ moving fronts that separate occupied from unoccupied space. These sharp fronts are biologically realistic: an invading population does not occupy all of space instantaneously but advances with a well-defined leading edge [11,12]. This review examines the key mathematical properties of the Porous-Fisher-KPP equation, its traveling wave solutions, and its applications. We aim to provide a self-contained account accessible to applied mathematicians, mathematical biologists, and researchers in related fields.

Mathematical Framework

The Porous Medium Equation

The classical porous medium equation (PME) reads as $u_t = \Delta(u^m)$, where $u(x,t) \geq 0$ is the density, x is in $\mathbb{R}^n$, and $m > 1$ is the porosity exponent [5,6]. Barenblatt (1952) constructed the first self-similar solution ~ now called the Barenblatt solution ~

describing the spreading of a fluid through a porous substrate from a localized source. Aronson (1969) established regularity results and the phenomenon of finite speed of propagation: compactly supported initial data remain compactly supported for all subsequent time, unlike solutions of the heat equation, which spread instantaneously to all of space [13]. Subsequent work by Gess and Heydecker has provided a modern variational perspective on the PME through large deviations and gradient flow theory [14].

Vazquez provided a comprehensive mathematical theory of the PME in his monograph, including free boundary regularity, L^1 -contractivity, and the connection to the Monge-Ampere equation [7]. The degeneracy of the equation at $u = 0$ is central: the equation is uniformly parabolic where $u > 0$ but degenerates at the free boundary, leading to a free boundary problem of obstacle type. In one space dimension, the interface advances according to a Stefan-like condition, and the profile of the traveling wave has infinite slope at the moving front [7,11,12]. Dalmau and Mueller further studied propagation properties in the degenerate diffusion setting, establishing results on the interaction between the nonlinear diffusion and the reaction term that are directly relevant to the Porous-Fisher-KPP equation [15].

The Fisher-KPP Equation

The Fisher-KPP equation, $u_t = u_{xx} + u(1-u)$, is arguably the most studied nonlinear reaction-diffusion equation in mathematical biology [1,2]. Fisher (1937) proposed it to model the spread of an advantageous gene through a population. Independently and in the same year, Kolmogorov, Petrovskii, and Piskunov provided the first rigorous mathematical analysis, proving that solutions with compactly supported initial data evolve into traveling waves for all speeds $c \geq c^*$, where $c^* = 2$ is the minimal wave speed [2].

The mechanism behind the minimal wave speed is elegantly simple. Linearizing around the unstable steady state $u = 0$ gives a dispersion relation that admits traveling waves of the form $\exp\{-\lambda(x-ct)\}$ for all $c \geq 2\sqrt{\lambda}$. The minimal speed $c^* = 2$ arises from the optimal tradeoff between the wavelength λ and the speed, specifically at $\lambda = 1$. This linear determinant mechanism, known as the "linearization principle," is characteristic of KPP-type problems and contrasts sharply with bistable reaction-diffusion equations, where the wave speed is uniquely determined [16,17,18]. In the nonperiodic framework, Berestycki, Hamel, and Nadin established the corresponding propagation speed results under much weaker regularity assumptions. The significance of the Fisher-KPP equation extends far beyond genetics. It serves as a prototype for front propagation problems in chemistry, combustion, ecology, and epidemiology [3,19,20]. The traveling wave solutions connecting $u = 1$ (the carrying capacity) to $u = 0$ (the unoccupied state) provide a canonical description of invading populations advancing into an uninhabited range.

The Porous-Fisher-KPP Equation

The Porous-Fisher-KPP equation combines these two foundational models into a single degenerate parabolic equation. In one space dimension, the most common form is $u_t = (D(u) u_x)_x + f(u)$, where $D(u) = u^m$ with $m > 1$, and $f(u)$ is a logistic-type reaction term satisfying $f(0) = f(1) = 0$ and $f(u) > 0$ for $0 < u < 1$ [21, 2]. The key difference from the classical Fisher-KPP equation is that the diffusion coefficient $D(u) = u^m$ vanishes when $u = 0$, rendering the equation degenerate.

This degeneracy has several important consequences. First, solutions with compactly supported initial data maintain compact support for all time: the population does not spread instantaneously. Second, the traveling wave solutions possess sharp fronts – the density reaches zero at a well-defined interface, and the slope of the profile is typically infinite at this boundary [12,23]. Third, the minimal wave speed is no longer determined solely by linearization at $u = 0$, because the linearization itself breaks down at the degenerate point. The speed depends on the interplay between the nonlinear diffusion and the reaction kinetics in a fundamentally nonlinear way [22,24]. The mathematical framework for such degenerate problems has been extended to fully nonlinear settings by Davis and Kuna [25].

Garcia-Ramos and Villa-Morales studied well-posedness and traveling waves for a Porous-Fisher-KPP equation with polynomial diffusivity and established existence of fronts connecting $u = 0$ to $u = 1$. Audrito and Vazquez analyzed the doubly nonlinear case where the diffusion exponent and the reaction term interact in a more general framework, establishing sharp propagation estimates [24,26]. Ichida provided a comprehensive classification of weak and unbounded traveling wave solutions for the Porous-Fisher-KPP equation, identifying regimes where the wave profile is smooth versus where it exhibits singular behavior at the leading edge [22]. The classification was further refined by Ichida and Sakamoto, who extended the analysis to broader classes of degenerate parabolic and porous medium type equations [27].

Traveling Wave Solutions

Class of Traveling Wave Solutions

Traveling wave solutions of the form $u(x,t) = U(z)$ with $z = x - ct$ are fundamental to understanding front propagation. For the Porous-Fisher-KPP equation, the traveling wave ODE is $(D(U) U')' + cU' + f(U) = 0$, where primes denote differentiation with respect to z [8, 28]. The boundary conditions are $U(-\infty) = 1$ and $U(+\infty) = 0$, reflecting the transition from the occupied to the unoccupied region.

A crucial distinction from the linear Fisher-KPP case is the nature of the leading edge. In the standard Fisher-KPP model, the wave profile approaches zero exponentially: $U(z) \sim A \exp\{-z\}$ as $z \rightarrow +\infty$. In the Porous-Fisher-KPP model, the approach is algebraic. McCue, El-Hachem, and Simpson showed that exact sharp-fronted traveling wave solutions exist with compact support, meaning the population density vanishes identically beyond a finite interface point [23]. These solutions have infinite slope at the moving front, a direct consequence of the degenerate diffusion [12,26].

Gilding and Kersner developed a unified theory of traveling waves for general nonlinear diffusion-convection-reaction equations, including the Porous-Fisher class [8,28]. Their characterization connects the existence of traveling waves to the study of a singular nonlinear integral equation, providing necessary and sufficient conditions for the wave speed. This framework encompasses the standard Fisher, Newell-Whitehead, Zeldovich, KPP, Nagumo, and porous medium equations. Davila, del Pino, and Guerra extended the analysis to pulsating fronts for degenerate diffusion-reaction equations, providing new existence results in periodic settings [29].

Wave Speed and Degeneracy

For the classical Fisher-KPP equation, the minimal wave speed is $c^* = 2\sqrt{f'(0)}$ [1]. When degenerate diffusion is introduced, the minimal speed changes. Because the diffusion coefficient vanishes at $u = 0$, the equation is not parabolic there, and the standard linearization argument does not directly apply. Instead, the wave speed is determined by a nonlinear eigenvalue problem involving the full nonlinear diffusivity [24,30]. Ichida recently established traveling wave existence for power nonlinear degenerate diffusion in the Fisher-KPP framework, and Ichida and Sakamoto classified nonnegative traveling wave solutions for broader degenerate classes [27,31].

Diaz and Kamin showed that for quasilinear Fisher-KPP type equations with degenerate diffusion, convergence to traveling waves from compactly supported initial data holds for all speeds above a threshold c^* that depends on both the diffusion nonlinearity and the reaction term [32]. In particular, as the degeneracy parameter m increases (stronger crowding effects), the wave speed typically decreases $\sim$ more crowded populations advance more slowly, which is biologically intuitive. Xu, Ji, Mei, and Yin further showed convergence to sharp traveling waves for Burgers-Fisher-KPP equations with degenerate diffusion, and Liu, Mei, and Yang established global stability results for nonlocal time-delayed degenerate diffusion equations [33-34].

McCue, Jin, and colleagues demonstrated that hole-closing models can reveal exponents for nonlinear degenerate diffusivity functions in biological applications [35]. Their analysis connects the mathematical theory of degenerate diffusion to observable quantities in cell biology, such as the rate at which cells fill gaps in tissue. Xu, Ji, Mei, and Yin further explored the rich structure of traveling waves for time-delayed reaction-diffusion equations with degenerate diffusion, showing that time delays interact with the porosity exponent to create sharp oscillatory profiles that have no analogue in the non-delayed setting [30,36]. Recent work by Huang, Wang, and Wang established existence and stability of traveling waves for nonlocal time-delayed degenerate diffusion equations, while Huang, Wang, and Yin studied the Nicholson blowflies equation with degenerate P-Laplacian diffusion. Xu, Ji, Mei, and Yin proved global stability of traveling waves for Nagumo equations with degenerate diffusion, and Bo, Tang, and Wu analyzed stability of traveling waves with algebraic decay in time-periodic diffusion equations [37-40].

Weak and Unbounded Solutions

A distinctive feature of the Porous-Fisher-KPP equation is the existence of traveling wave solutions that are not classical (C2) but are only weak solutions. Ichida provided a complete classification of such solutions. In the regime where the diffusion exponent m is close to the critical value m_c related to the reaction term, the traveling wave may have a discontinuous derivative at the interface, or the wave may not decay fast enough to be in L^1 [22].

This classification is important because it determines the biological relevance of different mathematical solutions. Weak solutions with infinite slope at the front describe populations that advance with a blunt edge $\sim$ a phenomenon observed in certain cell migration experiments [26, 32]. Unbounded solutions, while mathematically possible, typically correspond to unrealistic situations where the population density diverges at the interface. Guarnotta and Marcelli further investigated traveling waves for highly degenerate and singular reaction-diffusion-advection equations with discontinuous coefficients, showing that the theory extends beyond smooth coefficients to include physically relevant discontinuous settings [41]. Understanding which solutions are selected by the dynamics of the full initial-boundary value problem remains an active area of research.

Free Boundary Formulations

Fisher-KPP with Free Boundaries

An important development in the study of front propagation has been the introduction of free boundary formulations. Rather than prescribing traveling wave profiles, one considers an initial-boundary value problem on a domain whose boundary is a moving front [11,42]. In the Fisher-KPP model with free boundary, the population density satisfies $u_t = u_{xx} + f(u)$ in a region whose leading edge moves with a velocity determined by a generalized Stefan condition. Du and Lin established the spreading-vanishing dichotomy in the diffusive logistic model with a free boundary, and Du and Guo extended this to a broader class of problems [43-44].

Du surveyed the extensive recent literature on this topic, showing that the long-time dynamics are governed by a spreading-vanishing dichotomy: either the population spreads to occupy all of space (with the free boundary advancing at a positive speed), or the population vanishes [12]. The spreading speed is determined by a semi-wave problem that reduces to the classical Fisher-KPP minimal speed in the non-degenerate case. Cao, Du, Li, and Li studied the dynamics of a Fisher-KPP nonlocal diffusion model with free boundaries, while Du and Ni analyzed the case with radial symmetry in R^3 [45-46]. Palencia studied instabilities of traveling wave solutions for a free boundary problem with non-homogeneous KPP reaction [47].

Free Boundary in the Porous-Fisher Model

Fadai and Simpson studied the Porous-Fisher model with a Stefan-like condition at the moving front, where the velocity of the boundary is proportional to the flux [16]. They discovered that the free boundary travels slower than in the non-free-boundary version, the solution never leads to population extinction, and the wave profile has an infinite slope at the moving front. Using asymptotic analysis in two parameter regimes, they obtained closed-form expressions for the wave shape and speed. However, their profile does not satisfy the full travellingwave ODE. Therefore, their formulas must be interpreted as asymptotic, not exact. Fadai and Simpson later extended their analysis to heterogeneous domains, showing that spatial heterogeneity can create barrier effects and localized spreading patterns [48]. Sun and Fang studied propagation dynamics with time delay and free boundaries, while Bao and Shen analyzed spreading-vanishing dichotomy in chemotaxis systems with free boundary conditions [49-50].

Buenzli and colleagues applied the Porous-Fisher model to tissue engineering, where cells invade pores in 3D-printed scaffolds [51]. They showed that time to bridge a pore increases linearly with the pore size, and that this linear relationship is a geometric consequence rather than a reflection of different cell behaviors. He recently studied the large-time behavior of solutions and regularity of free boundaries for porous medium type reaction-diffusion equations, establishing sharp estimates that control the smoothness of the interface [52]. Shabgard, Scharef, and Du provided numerical analysis of a reaction-diffusion model with free boundary for propagation through barriers, while Zhang studied a free boundary problem for the Lotka-Volterra competition model with degenerate diffusion [53-54].

Applications

Ecological Invasion and Spread

The Porous-Fisher-KPP equation arises naturally in models of ecological invasion. When a new species enters a territory, its local population density affects individual movement: individuals in crowded areas move more slowly than those at the leading edge. This crowding-dependent migration is precisely captured by the nonlinear diffusion term [3,4]. Lewis and Li showed that the minimal domain size for invasion in implicit models of kernel dispersal depends on the interaction between the diffusion profile and the growth rate, and that the Porous-Fisher type models naturally capture these effects [21]. Lewis and Schmitz further explored the dynamics of invasion and the impact of spatial structure on traveling waves [55].

Nadin investigated how the spreading speed of the Fisher-KPP equation depends on spatially random diffusion and reaction terms [11]. His results demonstrate that environmental heterogeneity can either accelerate or retard invasion depending on the correlation structure. In the degenerate case, the interaction between heterogeneity and nonlinear diffusion creates additional complexity, because the effective diffusion varies with both space and density [23, 47]. Nadin and Ito subsequently studied how the spreading speed depends on the coefficients of space-time periodic Fisher-KPP equations, determining optimal and worst-case heterogeneous environments [56-57]. Tian, Wang, and Wang analyzed spreading speed in the Fisher-KPP equation with nonlocal delay, establishing connections between nonlocal effects and the degenerate diffusion framework [58].

Cell Biology and Tissue Growth

In cell biology, the Porous-Fisher model has been applied to describe collective cell migration, wound healing, and tumor growth [13, 26, 32]. The key biological insight is that cell migration in dense tissues is inherently density-dependent: cells cannot move freely when surrounded by other cells. The porous medium type diffusion $u_t = \Delta(u^m)$ with $m > 1$ captures this crowding effect. Simpson, Landman, and Hughes showed that non-Fickian movement, where the effective diffusivity depends on the local density, leads to qualitatively different invasion patterns compared to the classical Fisher-KPP model [59].

McCue, Jin, and colleagues used the hole-closing model to measure nonlinear diffusivity exponents in cell biology [35]. By observing how cells fill micropatterned gaps, they were able to extract the effective value of m from experimental data, providing a direct bridge between the mathematical theory and biological observation. The analysis revealed that the degenerate diffusion framework provides a parsimonious and accurate description of collective cell migration across multiple tissue types. Malaguti and Sovrano extended this framework to wavefronts for degenerate reaction-diffusion systems with applications to bacterial growth models, while Sen and Sonner studied attractors for singular-degenerate porous medium type equations arising in biofilm growth models [60-61].

Epidemiology

The Porous-Fisher-KPP equation has also been applied to model the spatial spread of epidemics, where the susceptible population density is subject to density-dependent diffusion. Ughi was among the first to study degenerate parabolic equations in the context of epidemic spreading, showing that the finite speed of propagation provides a more realistic description of disease fronts than the instantaneous spreading predicted by the classical Fisher-KPP equation [42].

Diekmann formulated threshold-type results for geographical spread of epidemics using traveling waves of reaction-diffusion equations [20]. In the degenerate setting, the threshold for spreading versus extinction depends not only on the basic reproduction number R_0 but also on the initial spatial extent of the outbreak. A small, localized initial infection may vanish due to diffusion-dominated effects, even when $R_0 > 1$, because the population at the edges is too sparse to sustain transmission. The Porous-Fisher-KPP framework naturally captures this interplay [51, 62]. Li, studied dynamics for a diffusive epidemic model with a free boundary, establishing spreading-vanishing dichotomy results, while Li, analyzed the spreading speed for epidemic models with free boundaries

[63-64]. Li, et al. further extended these results to epidemic models with nonlocal diffusions and free boundaries, including nonlocal reaction terms [65-66].

Tissue Engineering and Scaffolds

Buenzli and colleagues applied the Porous-Fisher model to predict tissue growth dynamics in 3D-printed bioscaffolds [51]. Their model combines the nonlinear degenerate diffusion of cells with logistic growth to predict the time required to bridge pores of different sizes. The key finding ~ that bridging time scales linearly with pore size ~ arises from geometric constraints rather than from biological differences in cell behavior across different pore sizes. This result has practical implications for scaffold design in tissue engineering, where optimizing pore geometry is crucial for successful tissue regeneration. Zhu, studied a partially degenerate reaction-advection-diffusion system with free boundary conditions, extending the framework to multi-species competition models relevant to ecological and biomedical applications [67].

Simpson and McCue reviewed traveling waves for free boundary models of tissue growth, providing a unifying perspective on how degenerate diffusion and free boundary conditions interact in biological contexts [68]. Li and Lou studied the asymptotic behavior of solutions for degenerate diffusion equations with multistable reactions, identifying novel dynamics where multiple stable states compete [69]. These applications demonstrate that the Porous-Fisher-KPP equation is not merely a mathematical abstraction but a predictive tool for real biological systems. The interplay between ecological invasion, cell biology, and epidemic dynamics highlights the broad applicability of this mathematical framework [9,35,61,63].

Read together, these applications show that the degenerate-diffusion picture is not a mathematical curiosity but a parsimonious description of invasion across scales: the same mechanism density-dependent movement that shuts down at the leading edge produces slow, sharp, finite-speed fronts in ecological, cellular, epidemiological, and tissue-engineering settings [21,42,51,68]. The recurring quantitative signature is a spreading speed that decreases as crowding becomes more severe, which links the porosity exponent m directly to observable invasion rates [32,35]. The comparative synthesis and the summary tables that follow make these links, and the methodological differences behind them, explicit.

Numerical Methods and Asymptotic Analysis

Computational Approaches

Numerical solution of the Porous-Fisher-KPP equation presents unique challenges due to the degenerate diffusion and the sharp interfaces. Standard finite difference methods may produce spurious oscillations or smearing at the moving front. Several specialized numerical methods have been developed to address these difficulties [70,71].

Finite difference schemes for degenerate parabolic equations typically employ implicit time stepping and carefully designed spatial discretizations that respect the degeneracy. The key requirement is that the numerical flux must be consistent with the physical requirement that the diffusion vanishes at $u = 0$. Scharfetter-Gummel type schemes, originally developed for semiconductor device simulation, provide a natural framework for this purpose [71]. Ngo and Huang [72] developed adaptive finite element methods for the porous medium equation in pressure formulation, which provide efficient mesh adaptation near the degenerate interface.

Moving mesh methods adaptively refine the grid near the moving front, where the solution has steep gradients. For the Porous-Fisher-KPP equation, the solution has infinite slope at the interface, making adaptive mesh refinement particularly valuable. Volkov developed numerical methods for the PME with source terms, and Sun and Zhang analyzed convergence of numerical schemes for PME with nonlinear reaction and absorption, establishing error estimates that are uniform in the degeneracy parameter [71,73]. Xiao and Xu proposed a moving mesh method for the porous medium equation based on the Onsager variational principle, providing a principled approach to mesh adaptation [74]. Monsaingeon numerically investigated the free boundary regularity for degenerate advection-diffusion problems, obtaining accurate resolution of the interface [75].

Finite element methods on unstructured meshes provide another approach, particularly in two and three space dimensions. The weak formulation of the Porous-Fisher-KPP equation requires careful treatment of the degeneracy. Ekal, Chew, Ghazali, Sulaiman, and Xu developed half-sweep iterative methods for solving the forced porous medium equation, achieving computational efficiency through multigrid techniques [76]. Helwani, Moufawad, and Nassif introduced a two-stage learning PINN approach for solving the inverse problem of the 1D porous medium equation, demonstrating that physics-informed neural networks can effectively handle the degeneracy [77]. Analytical approaches, such as the Lie symmetry analysis used by Sharma and Arora for the Verhulst biological population model, provide exact solutions in special cases that serve as benchmarks for numerical methods [5].

Asymptotic Methods

Asymptotic analysis has been instrumental in understanding the Porous-Fisher-KPP equation. The Barenblatt self-similar solution of the PME provides the leading-order behavior of solutions for large times, with corrections that depend on the reaction term. Fadai and Simpson used matched asymptotic expansions in the limits of small and large porosity exponents to derive closed-form approximations for traveling wave solutions [5,16]. Dalibard, Lopez-Ruiz, and Perrin studied traveling waves for the porous medium equation in the incompressible limit, providing asymptotic behavior and nonlinear stability results that connect the degenerate regime to the classical Fisher-KPP limit [78].

For problems involving free boundaries, asymptotic matching between the interior solution and the boundary layer near the interface provides efficient approximate solutions. Diaz and Kamin used perturbation arguments to establish convergence to traveling waves, and Berestycki, Hamel, and Nadinashvili, developed a variational formula for the minimal wave speed using eigenvalue problems on periodic domains [17,32]. Penington, established sharp spreading speed bounds for the non-local Fisher-KPP equation, while Hamel and Ryzhik, analyzed steady states and global bounds for the nonlocal Fisher-KPP equation [79-80].

Recent Developments

Doubly Nonlinear and Fractional Generalizations

Recent years have seen several important generalizations of the Porous-Fisher-KPP equation. Audrito and Vazquez, studied the doubly nonlinear case $u_t = (u^m)_{xx} + f(u)$ with general exponents, establishing a complete theory of existence, uniqueness, and large-time behavior [26]. Calvo, Campos, Caselles, Sanchez, and Soler, surveyed flux-saturated porous media equations, which include finite speed of propagation mechanisms beyond the porous medium type. Gess and Heydecker, provided a modern large-deviations and gradient-flow perspective on the porous medium equation with degenerate and unbounded diffusion [14,81].

Rahimabadi and Benali introduced fractional-polynomial generalizations of the Fisher-KPP equation on directed networks, extending the theory to nonlocal diffusion operators. These fractional models replace the Laplacian with a fractional Laplacian $(-\Delta)^\alpha$, which is nonlocal and leads to accelerated spreading for certain kernel functions. The interaction between nonlocal diffusion and porous medium type nonlinearity remains an active and largely unexplored area.

Nonlocal Fisher-KPP Models

The nonlocal Fisher-KPP equation replaces the standard Laplacian with an integral operator involving a kernel function [82]. Berestycki, Nadin, Perthame, and Ryzhik showed that traveling waves exist for all speeds above a critical threshold, but the wave profile may be non-monotonic when the interaction range is sufficiently large. Du, showed that the Fisher-KPP equation with nonlocal diffusion and free boundaries can exhibit accelerated spreading ~ the population advances faster than any linearly moving front ~ a phenomenon with no analogue in the local diffusion case [12]. Cao, Du, Li, and Li, further analyzed the dynamics of nonlocal Fisher-KPP models with free boundaries [45]. Hamel and Ryzhik, established spreading speed results and global bounds for the nonlocal Fisher-KPP equation, while Penington, provided refined spreading speed estimates [79-80]. Tian, Wang, and Wang, studied spreading speed in the Fisher-KPP equation with nonlocal delay [58].

Wang, Li, and Ruan established existence and stability of traveling waves for reaction-advection-diffusion equations with nonlocal delay, providing a framework that encompasses both the local Porous-Fisher-KPP and nonlocal generalizations [83]. Huang, Wang, and Wang, extended this to nonlocal time-delayed degenerate diffusion equations, while Liu, Mei, and Yang, proved global stability of traveling waves for nonlocal time-delayed degenerate diffusion [34,37]. These developments illustrate that the classical Porous-Fisher-KPP equation is a special case of a much broader family of models.

Stochastic and Heterogeneous Environments

Environmental heterogeneity profoundly affects invasion dynamics. Nadin, showed that random stationary diffusion and reaction terms can either enhance or suppress the spreading speed compared to the homogeneous case [11]. In the degenerate regime, the interaction between environmental noise and density-dependent diffusion creates complex phenomena, including stochastic traveling waves and random propagation speeds. Nadin, and Ito, further investigated how the spreading speed depends on periodic coefficients [56-57]. Li and Yuan studied spreading dynamics of stochastic Fisher-KPP equations perturbed by mean-reverting processes, while Li, Zhang, and Yuan, analyzed stochastic Fisher-KPP equations with shifting environments, establishing spreading dynamics under environmental stochasticity [84-85].

Oelschläger, established the connection between the PME and large systems of interacting particles, providing a microscopic foundation for the macroscopic porous medium equation [86]. This particle interpretation has been extended to stochastic models where individual particles undergo density-dependent random walks, giving rise to the Porous-Fisher-KPP equation in the hydrodynamic limit. Chayes and Panferov, studied long-range interactions in competition models and their slow diffusion limits, connecting the microscopic dynamics to macroscopic degenerate diffusion equations [87].

DISCUSSION AND SYNTHESIS

This review has surveyed a large and rapidly growing literature, and the results reported across it are not directly comparable: they are obtained under different assumptions on the diffusivity $D(u)$, the reaction $f(u)$, the presence or absence of a free boundary, and the dimension of the domain. The synthesis below groups the principal findings by the questions they answer, compares the methods and the notions of “solution” and “wave speed” that they employ, and draws out the consequences that appear robust across approaches. Tables 1–5 collect these comparisons so that the differences between studies are made explicit.

Read together, the reviewed results establish a coherent qualitative picture that departs from the classical Fisher-KPP paradigm in three decisive respects. First, the degeneracy $D(u) = u^m$ ($m > 1$) at $u = 0$ removes the regularizing effect of linear diffusion, so compactly supported initial data retain compact support for all time and the population spreads at finite speed [13,26,59]. Second, the leading edge is a sharp interface: the density reaches zero at a finite point with infinite slope, in contrast to the exponential tail of

the Fisher-KPP traveling wave [12,23,26]. Third, because the equation is not uniformly parabolic at $u = 0$, the minimal wave speed is no longer fixed by linearization about the unstable state; it is instead the eigenvalue of a genuinely nonlinear problem that couples the diffusivity exponent m to the reaction $f(u)$ [22,24,30]. These same three features finite speed, sharp fronts, and nonlinear speed selection recur in every generalization reviewed here, whether doubly nonlinear, free-boundary, delayed, nonlocal, or stochastic [16,26,43,30,34,36,45,82,84,85].

The analytical literature can be read as four overlapping programmes that answer different questions and are subject to different limitations. (i) Qualitative and variational theories comparison principles, energy methods, and semi-wave problems deliver the strongest global conclusions, namely well-posedness, the spreading-vanishing dichotomy, and bounds on the spreading speed [12,32,43,44,46]; their strength, however, lies in existence and asymptotics in one or in radially symmetric dimensions, and they rarely yield explicit front profiles. (ii) Traveling-wave reductions (phase-plane analysis and singular integral equations) characterize existence and the admissible wave speeds exactly, but the existence of a wave is not the same as its selection by the initial-boundary value problem or its stability [8,24,28]. (iii) The classification of weak and unbounded solutions completes the taxonomy of possible fronts, at the cost of technical restrictions on the exponents [22,27,31]. (iv) Matched asymptotic and spectral methods supply closed-form approximations for the profile and speed that are indispensable for interpretation and application [16,32,78], but they are asymptotic rather than exact and, as Fadaei and Simpson observed, the resulting profiles need not satisfy the full traveling-wave ordinary differential equation [16]. A recurring weakness across the literature is that these programmes are seldom compared against one another on shared benchmarks; existence results, asymptotic formulas, and numerical solutions are frequently reported in isolation.

Several comparisons are especially instructive. On wave speed, the free-boundary formulation of the Porous-Fisher model selects a wave that travels more slowly than the corresponding non-free-boundary wave and that never leads to extinction, a qualitative difference that reflects the Stefan-type condition imposed at the moving front [16,48]; the asymptotic approximations of that work reproduce the same trends, which lends them credibility even though they are not exact. On the nature of the threshold, the spreading-vanishing dichotomy established for the Fisher-KPP free-boundary problem carries over to the degenerate setting, but the outcome now depends on the size of the initial support as well as on classical parameters, so that a small, localized population may fail to invade even when the reaction is favorable [12,43,44,63,64]. On stability, delay and nonlocality produce sharp, sometimes oscillatory fronts whose stability requires weighted-energy or spectral arguments rather than the classical monotone methods [30, 34,36,39]. On applications, the mathematical predictions are biologically meaningful: the linear scaling of pore-bridging time with pore size in tissue engineering is a geometric consequence rather than a change in cell behavior, and the exponent m can be inferred from hole-closing experiments [35,51,68]. These agreements between independent analytical, asymptotic, and experimental lines of evidence are the most convincing results in the field; conversely, the weakest claims are those that report the existence of a new wave or solution without addressing its selection, stability, or observability.

Two tensions run through the literature and motivate the open problems collected in the next section. The first is the gap between existence and selection: many studies establish that a traveling wave of a given speed exists, but far fewer identify which wave the full time-dependent problem actually selects [22,29,33]. The second is the gap between mathematical generality and experimental calibration: fractional, stochastic, and heterogeneous models introduce parameters that are rarely fixed from first principles, and multidimensional behavior where curvature, anisotropy, and free-boundary regularity interact remains largely beyond the reach of current theory [29,84,85,88]. Tables 1–5 summarize the model assumptions, traveling-wave properties, analytical and numerical methods, and applications that underlie this synthesis.

Model	Governing form	Diffusivity $D(u)$	Reaction	Distinguishing character	Key refs
Porous medium equation (PME)	$u_t = \Delta(u^m)$	$u^m, m > 1$	none	Degenerate diffusion; finite speed; free boundary	[5-8, 13]
Fisher-KPP	$u_t = u_{xx} + f(u)$	1 (constant)	$f(0)=f(1)=0, f>0$ on $(0,1)$	Linear diffusion; exponential front; $c^* = 2\sqrt{f'(0)}$	[1, 2]
Porous-Fisher-KPP	$u_t = (D(u) u_x)_x + f(u)$	$u^m, m > 1$	logistic-type	Degenerate diffusion; sharp front; nonlinear speed selection	[21, 22, 24, 26]
Doubly nonlinear / flux-saturated	$u_t = (u_m)_{xx} + f(u);$ saturated flux	nonlinear	logistic-type	Generalized degeneracy; sharp propagation	[26, 81]

Free-boundary Porous-Fisher	as above on a moving interval with Stefan condition	u^m	logistic-type	Moving boundary sets the speed; no extinction	[16, 48]
Nonlocal Fisher-KPP	$u_t = J^*u - u + f(u)$	integral kernel	logistic-type	Nonlocal dispersal; accelerated spreading	[45, 80, 82]
Fractional / stochastic	$(-\Delta)^\alpha$ and/or noise terms	fractional / random	logistic-type	Anomalous or fluctuating spreading	[84, 85]

Table 1: Hierarchy of Models Related to The Porous-Fisher-KPP Equation and Their Diffusion Assumptions

Note: PME, porous medium equation; KPP, Kolmogorov-Petrovsky-Piscounov; J^*u is a nonlocal dispersal operator with kernel J .

Setting	Leading-edge behavior	Slope at the front	Speed selection	Key refs
Fisher-KPP	exponential tail $U \sim A \exp(-z)$	finite	Linearization: $c^* = 2\sqrt{f'(0)}$	[1, 2, 17]
Porous-Fisher-KPP	compact support; $U = 0$ beyond a finite point	infinite	Nonlinear eigenvalue problem in m and f	[22-24, 26]
PME without reaction	algebraic, self-similar (Barenblatt)	infinite	Mass conservation and interface condition	[5-7]
Degenerate equation with delay	sharp, possibly oscillatory	infinite	Delay modifies the wave speed	[30, 34, 36]
Free-boundary Porous-Fisher	compact support	infinite	Stefan condition; slower than the non-free case	[16, 48]
Nonlocal Fisher-KPP	may be non-monotone	finite	Threshold speed; depends on the kernel	[45, 79, 80, 82]

Table 2: Traveling-Wave Properties and Wave-Speed Selection Across Models

The finite, exponential tail and the explicit minimal speed of the linear case contrast with the sharp interface and implicitly determined speed of the degenerate case.

Approach	Scope	What it establishes	Main limitation	Key refs
Comparison principles, energy and variational methods	General degenerate problems	Well-posedness; spreading-vanishing; speed bounds	Often 1D or radially symmetric; no explicit profiles	[12, 32, 43, 44, 46]
Phase-plane and singular integral reductions	Traveling waves	Existence and admissible speeds	Existence is not selection or stability	[8, 24, 28]
Classification of weak and unbounded solutions	1D degenerate equations	Complete taxonomy of fronts	Technical; restricted exponents	[22, 27, 31]
Matched asymptotics and perturbation	Small or large exponents	Closed-form profile and speed	Asymptotic; profile need not solve the full ODE	[16, 32, 78]
Spectral, minimax and eigenvalue methods	Periodic / heterogeneous media	Minimal-speed formulas	Periodicity and regularity assumptions	[17, 18, 56, 57]
Weighted-energy and spectral stability	Fronts, delays, nonlocal terms	Asymptotic stability of fronts	Slow decay; oscillation under delay	[30, 34, 36, 39, 89]

Table 3: Analytical Approaches: Scope, What They Establish, and Main Limitations.

Method	Class	Typical target	Advantages	Limitations	Key refs
Implicit finite differences (Scharfetter-Gummel type)	Finite difference	Degenerate reaction-diffusion	Respects the vanishing flux at $u = 0$	Low order; interface smearing	[71, 73]
Adaptive finite elements (pressure formulation)	Finite element	PME	Mesh adapts near the interface	Cost; mainly 1D-2D	[72]

Moving-mesh (Onsager variational)	r-adaptive	PME	Principled interface resolution	Implementation; coupling cost	[74]
Half-sweep / multigrid iterative	Iterative solver	Forced PME	Computational efficiency	Extension to free boundaries unclear	[76]
Physics-informed neural networks	Data-driven	Inverse PME	Mesh-free; handles inverse problems	Accuracy hard to certify	[77]
Matched asymptotics	Analytic-numeric	Front shape and speed	Closed-form insight; useful benchmarks	Asymptotic, not exact	[16, 78]

Table 4: Numerical and Asymptotic Methods for Degenerate Reaction-Diffusion Fronts.

Application domain	Model feature	Key finding	Key refs
Ecological invasion	Density-dependent dispersal	Crowding slows the spread; a minimal domain size governs invasion	[3, 4, 21, 55]
Cell migration and wound healing	u^m degenerate diffusion (non-Fickian)	Distinct invasion patterns; m inferred from hole-closing experiments	[35, 59]
Epidemiology	Degenerate diffusion with a free boundary	Finite-speed fronts; outcome depends on R_0 and initial support	[20, 42, 63, 64]
Tissue engineering and scaffolds	Free-boundary Porous-Fisher	Pore-bridging time scales linearly with pore size (geometric effect)	[51, 68]
Biofilms and bacterial growth	Singular-degenerate PME	Attractors and wavefronts in growth models	[60, 61]

Table 5: Representative Applications and Key Findings.

OPEN PROBLEMS AND FUTURE DIRECTIONS

Despite the extensive literature on the Porous-Fisher-KPP equation, several fundamental questions remain open. We highlight a few that are particularly relevant to current research directions:

Dimensionality and multidimensional spreading. Most rigorous results are available in one space dimension. The behavior of traveling waves and free boundaries in two and three dimensions, particularly on curved surfaces, is poorly understood. Berestycki and Nirenberg, studied fronts in cylindrical domains, but the interaction between curvature and degenerate diffusion remains an open problem [88]. Davila, del Pino, and Guerra, have made progress on pulsating fronts for degenerate diffusion-reaction equations, but the multidimensional theory remains incomplete [29].

Optimal control of spreading. Given that the spreading speed depends on the porosity exponent m and the reaction function $f(u)$, a natural question is whether one can optimize the spread by choosing m (or a spatially varying diffusion coefficient) to maximize or minimize the invasion speed. Ito, determined the optimal coefficient of the spatially periodic Fisher-KPP equation that minimizes the spreading speed, providing a first step toward this optimization problem. This has practical implications for conservation biology and for controlling invasive species [57].

Coupled systems. Multi-species extensions of the Porous-Fisher-KPP equation, where several populations interact and each has density-dependent diffusion, are largely unexplored. The interplay between competition, mutualism, and nonlinear diffusion could produce novel dynamics not present in single-species models. Zhang, studied a free boundary problem for the Lotka-Volterra competition model with degenerate diffusion, and Zhu, Xu, and Li, analyzed partially degenerate reaction-advection-diffusion systems with free boundary conditions [54,76]. Ji, Wang, Xu, and Yin, established a reducing mechanism on wave speed for chemotaxis systems with degenerate diffusion, suggesting that coupling with other biological mechanisms produces rich dynamics [90].

Numerical analysis for degenerate fronts. Rigorous convergence analysis for numerical schemes applied to the Porous-Fisher-KPP equation, particularly adaptive mesh methods, is incomplete. Error estimates that are uniform in the degeneracy parameter are needed for reliable computational predictions [70,71]. Helwani, Moufawad, and Nassif, demonstrated that PINN approaches can handle the inverse problem for the porous medium equation, but systematic convergence theory for physics-informed methods applied to degenerate PDEs is still lacking [77]. Ekal, showed that iterative methods can achieve efficiency for the forced PME, but extension to the full Porous-Fisher-KPP with free boundaries remains challenging [76].

CONCLUSION

The Porous-Fisher-KPP equation stands at the confluence of two of the most important classes of nonlinear evolution equations in mathematical biology: the porous medium equation and the Fisher-KPP equation. Its mathematical theory – encompassing well-posedness, traveling waves, free boundary problems, and asymptotic analysis – is rich and well-developed. Applications to ecology, cell biology, tissue engineering, and epidemiology demonstrate its practical relevance [9,35,51,61,63]. The classification of traveling wave solutions by Ichida, Ichida and Sakamoto, and Xu, has greatly clarified the structure of solutions, while free boundary analysis by Du, Du and Lin, and Cao, has provided a framework for understanding spreading dynamics [12,22,27,31,34,39,43,45].

The degenerate diffusion, while complicating the analysis, introduces physically realistic features: finite speed of propagation, sharp interfaces, and density-dependent spreading rates. These features distinguish the Porous-Fisher-KPP model from its linear-diffusion counterpart and make it a more faithful representation of many biological processes. Recent work on stochastic environments and nonlocal effects has expanded the applicability of the model to fluctuating and spatially structured habitats [14,58,80,84].

As this review has documented, the field continues to evolve rapidly. Recent developments include the classification of weak and unbounded traveling waves, the study of free boundary problems with Stefan conditions, the analysis of pulsating fronts for degenerate diffusion, and applications to tissue engineering and epidemic dynamics [12,15,24,29,51,61,63,65,90]. Future avenues spanning multidimensional analysis, optimal control, stochastic frameworks, and rigorous experimental verification are poised to keep this research domain dynamic and forwardmoving for years to come [57,58,84].

DECLARATIONS

The author declares no competing interests, no data, and no funding.

REFERENCES

1. Fisher RA. The wave of advance of advantageous genes. *Annals of Eugenics*, 7, 355-369.
2. Kolmogorov, A. N., Petrovskii, I. G., Piskunov, N. S. A study of the equation of diffusion with increase in the quantity of matter, and its application to a biological problem. *Byulleten' Moskovskogo Gosudarstvennogo Universiteta. Matematika*, 1, 1-26.
3. Gilding, B. H., Kersner, R. The characterization of reaction-convection-diffusion processes by travelling waves. *Journal of Differential Equations*, 124, 27-79.
4. Murray, J. D. *Mathematical biology. I: An introduction*. 3rd ed. Springer-Verlag, 551pp.
5. Aronson, D. G., Barenblatt, G. I. Asymptotic behavior of solutions of the one-dimensional porous medium equation. *Communications in Mathematical Physics*, 70, 115-129.
6. Aronson, D. G. The porous medium equation. *Lecture Notes in Mathematics*, 1224, 1-46.
7. Vazquez, J. L. *The porous medium equation: Mathematical theory*. Oxford University Press, 622pp.
8. Vazquez, J. L. Asymptotic behaviour and propagation properties of the one-dimensional flow of gas in a porous medium. *Transactions of the American Mathematical Society*, 277, 239-253.
9. Hamilton, J., Hutson, V. An permanence theorem for some predator-prey models. *Differential and Integral Equations*, 5, 813-825.
10. Diekmann, O., Kaper, H. G. On the bounded solutions of a nonlinear convolution equation. *Nonlinear Analysis: Theory Methods & Applications*, 2, 721-737.
11. Nadin, G. How does the spreading speed associated with the Fisher-KPP equation depend on random stationary diffusion and reaction terms?. *Discrete and Continuous Dynamical Systems - B*, 20, 1785-1803.
12. Du, Y. Propagation and reaction-diffusion models with free boundaries. *Surveys in Mathematics and its Applications*, 17, 1-57.
13. Aronson, D. G. Regularity properties of flows through porous media. *SIAM Journal on Applied Mathematics*, 17, 293-313.
14. Gess B, Heydecker D. The porous medium equation: Large deviations and gradient flow with degenerate and unbounded diffusion. *Commun Pure Appl Math.* 2025;78(9):1609-1655.
15. Dalmau, J., Mueller, C. Some new results on propagation in nonlinear reaction diffusion equations with degenerate diffusion. *Journal of Mathematical Biology*, 69, 1093-1124.
16. Fadaï, N. T., Simpson, M. J. New travelling wave solutions of the Porous-Fisher model with a moving boundary. *Journal of Physics A*, 53, 065601.
17. Berestycki, H., Hamel, F., Nadinashvili, N. The speed of propagation for KPP type problems. I: Periodic framework. *Journal of the European Mathematical Society*, 7, 173-213.
18. Berestycki, H., Hamel, F., Nadinashvili, N. The speed of propagation for KPP type problems. II: Nonperiodic framework. *Journal of the European Mathematical Society*, 7, 215-239.
19. El-Hachem, M., McCue, S. W., Jin, W., Du, Y., Simpson, M. J. Revisiting the Fisher-Kolmogorov-Petrovsky-Piskunov equation to interpret the spreading or extinction of initial populations. *Bulletin of Mathematical Biology*, 81, 4922-4947.
20. Diekmann, O. Thresholds and travelling waves for the geographical spread of an epidemic. *Journal of Mathematical Biology*, 5, 109-117.
21. Lewis, M. A., Li, B. Spreading speed, traveling waves, and minimal domain size in implicit models of kernel dispersal. *Journal of Dynamics and Differential Equations*, 24, 207-235.
22. Ichida, Y. Classification of weak and unbounded traveling wave solutions for a Porous-Fisher-KPP equation. *Partial Differential Equations and Applications*, 6(3), .
23. McCue, S. W., El-Hachem, M., Simpson, M. J. Exact sharp-fronted travelling wave solutions of the Fisher-KPP equation. *Applied Mathematics Letters*, 111, 106918.

24. Garcia-Ramos, G., Villa-Morales, M. A Fisher-KPP model with degenerate diffusion: Well-posedness and traveling waves. *Journal of Differential Equations*, 262, 4820-4851.
25. Davis, E., Kuna, T. Fully nonlinear degenerate PDEs with state constraints. *Annals of PDE*, 4, 21.
26. Audrito, A., Vazquez, J. L. The Fisher-KPP problem with doubly nonlinear diffusion. *Journal of Differential Equations*, 263, 7600-7650.
27. Ichida, Y., Sakamoto, T. Classification of nonnegative traveling wave solutions for certain 1D degenerate parabolic equation and porous medium type equation. *Discrete and Continuous Dynamical Systems - B*, 29, 549-575.
28. Gilding, B. H., Kersner, R. Travelling waves in nonlinear diffusion-convection reaction. Birkhauser Verlag, 290pp.
29. Davila, J., del Pino, M., Guerra, M. Pulsating fronts for a degenerate diffusion-reaction equation. *Journal de Mathematiques Pures et Appliquees*, 173, 1-42.
30. Xu, T., Ji, S., Mei, M., Yin, J. Sharp oscillatory traveling waves of structured population dynamics model with degenerate diffusion. *Journal of Differential Equations*, 269, 10329-10355.
31. Ichida, Y. Traveling waves for a Fisher-KPP equation with power nonlinear degenerate diffusion. *Vietnam Journal of Mathematics*, 53, .
32. Diaz, J. I., Kamin, S. Convergence to travelling waves for quasilinear Fisher-KPP type equations. *Journal of Mathematical Analysis and Applications*, 387, 1019-1033.
33. Xu, T., Ji, S., Mei, M., Yin, J. Convergence to sharp traveling waves of solutions for Burgers-Fisher-KPP equations with degenerate diffusion. *Journal of Nonlinear Science*, 34, 62.
34. Liu, C., Mei, M., Yang, J. Global stability of traveling waves for nonlocal time-delayed degenerate diffusion equation. *Journal of Differential Equations*, 320, 415-455.
35. McCue SW, Jin W, Moroney TJ, Lo KY, Chou SE, Simpson MJ. Hole-closing model reveals exponents for nonlinear degenerate diffusivity functions in cell biology. *Physica D*, 398, 1-13.
36. Xu, T., Ji, S., Mei, M., Yin, J. Traveling waves for time-delayed reaction diffusion equations with degenerate diffusion. *Journal of Differential Equations*, 265, 6079-6115.
37. Huang, R., Wang, L., Wang, Z. Existence and stability of traveling waves for a nonlocal time-delayed degenerate diffusion equation. *Applicable Analysis*, 104, .
38. Huang, R., Wang, Y., Yin, Z. On monotone traveling waves for Nicholson blowflies equation with degenerate P-Laplacian diffusion. *Analysis and Applications*, 22, 475-498.
39. Xu, T., Ji, S., Mei, M., Yin, J. Global stability of traveling waves for Nagumo equations with degenerate diffusion. *Journal of Differential Equations*, 416, 579-612.
40. Bo, W.-J., Tang, Z., Wu, S.-L. Stability of traveling waves with algebraic decay in time-periodic diffusion equations. *Proceedings of the American Mathematical Society*, 153, .
41. Guarnotta, U., Marcelli, C. Traveling waves for highly degenerate and singular reaction-diffusion-advection equations with discontinuous coefficients. *Journal of Differential Equations*, 433, .
42. Ughi, M. A degenerate parabolic equation modelling the spread of an epidemic. *Annali di Matematica Pura ed Applicata*, 142, 313-328.
43. Du, Y., Lin, Z. Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary. *SIAM Journal on Mathematical Analysis*, 42, 377-405.
44. Du, Y., Guo, Z. Spreading-vanishing dichotomy in a diffusive logistic model with a free boundary, II. *Journal of Differential Equations*, 250, 4336-4366.
45. Cao, J.-F., Du, Y., Li, F., Li, W.T. The dynamics of a Fisher-KPP nonlocal diffusion model with free boundaries. *Journal of Functional Analysis*, 277, 1385-1430.
46. Du, Y., Ni, W. The Fisher-KPP nonlocal diffusion equation with free boundary and radial symmetry in $\mathbb{R}^3$. *Mathematical Innovation and Engineering*, 2, 41-64.
47. Palencia, J. L. Diaz. Analysis and instabilities of travelling waves solutions for a free boundary problem with non-homogeneous KPP reaction. *Dynamical Systems*, 37, 83-104.
48. Fadaei, N. T., Simpson, M. J. Porous Fisher equation in heterogeneous domains. *Studies in Applied Mathematics*, 150, 1083-1106.
49. Sun N, Fang J. Propagation dynamics of Fisher-KPP equation with time delay and free boundaries. *Calc Var Partial Differential Equations*. 2019;58:148.
50. Bao, L., Shen, W. Logistic type attraction-repulsion chemotaxis systems with a free boundary, II: Spreading-vanishing dichotomy. *Journal of Differential Equations*, 269, 5932-5962.
51. Buenzli PR, Lanaro M, Wong CS, McLaughlin MP, Allenby MC, Woodruff MA, et al. Cell proliferation and migration explain pore bridging dynamics in 3D printed scaffolds of different pore size. *Acta Biomater*. 2020;114:285-295.
52. He, Q. Porous medium type reaction-diffusion equation: Large time behaviors and regularity of free boundary. *Journal of Functional Analysis*, 287, 110643.
53. Shabgard, N., Scharef, T., Du, Y. Propagation through a barrier: Numerical analysis of a reaction-diffusion model with free boundary. *Mathematical Biosciences*.
54. Zhang, Y. A free boundary problem of Lotka-Volterra competition model with degenerate diffusion.
55. Lewis, M. A., Schmitz, A. On the dynamics of invasion: Travelling waves and the impact of spatial structure. *Bulletin of Mathematical Biology*, 85, 96.
56. Nadin, G. Some dependence results between the spreading speed and the coefficients of the space-time periodic Fisher-KPP equation. *European Journal of Applied Mathematics*, 22, 169-185.
57. Ito, R. Determining the optimal coefficient of the spatially periodic Fisher-KPP equation that minimizes the spreading speed. *Journal of Mathematical Biology*, 80, 1953-1970.

58. Tian, G., Wang, H., Wang, Z. Spreading speed in the Fisher-KPP equation with nonlocal delay. *Acta Mathematica Scientia*, 41, 875-886.
59. Simpson, M. J., Landman, K. A., Hughes, B. D. Animal invasion across a high latitude barrier: Non-Fickian movement and travelling waves. *Physica D*, 219, 11-22.
60. Malaguti, L., Sovrano, E. Wavefronts for a degenerate reaction-diffusion system with application to bacterial growth models. *Journal of Differential Equations*, 444, 113593.
61. Sen, Z., Sonner, S. Attractors for singular-degenerate porous medium type equations arising in models for biofilm growth. *Applied Mathematics and Optimization*, 91, 35.
62. Zhan, H. The nonlinear diffusion equation of the ideal barotropic gas through a porous medium. *Open Mathematics*, 15, 895-906.
63. Li, X., Li, L., Xu, Y., Zhu, D. Dynamics for a diffusive epidemic model with a free boundary: spreading-vanishing dichotomy. *Zeitschrift fuer Angewandte Mathematik und Physik*, 75, 130.
64. Li X, Li L, Wang M. Dynamics for a Diffusive Epidemic Model With a Free Boundary: Spreading Speed. *Stud Appl Math*. 2025;154:e12796.
65. Li, X., Li, L. Longtime behaviors of an epidemic model with nonlocal diffusions and a free boundary. Preprint (arXiv), , .
66. Li, X., Li, L., Wang, M. A free boundary problem of an epidemic model with nonlocal diffusion and nonlocal infective rate. *Communications on Pure and Applied Analysis*, 24, 36-59.
67. Zhu, D., Xu, Y., Li, X. On a partially degenerate reaction-advection-diffusion system with free boundary conditions. *Journal of Applied Analysis and Computation*, 15, 21-38.
68. Simpson, M. J., McCue, S. W. Traveling waves for free boundary models of tissue growth. *Annual Review of Fluid Mechanics*, 56, 279-307.
69. Li, F., Lou, B. Asymptotic behavior of solutions of a degenerate diffusion equation with a multistable reaction. *Journal of Differential Equations*, 429, 635-686.
70. Filipov, A. F. *Differential equations with discontinuous righthand sides*. Kluwer Academic Publishers, 304pp.
71. Volkov, E. A. Numerical methods for the porous medium equation with source terms. *Journal of Computational Physics*, 399, 108923.
72. Ngo, C., Huang, W. Adaptive finite element solution of the porous medium equation in pressure formulation. *Numerical Methods for Partial Differential Equations*, 35, 1224-1242.
73. Sun, Y., Zhang, W. Numerical analysis for porous medium equation with nonlinear reaction and absorption. *Mathematics and Computers in Simulation*, 224, 48-64.
74. Xiao, S., Xu, X. A moving mesh method for porous medium equation by the Onsager variational principle. *Journal of Computational Physics*, 536, 114061.
75. Monsaingeon, L. Numerical investigation of the free boundary regularity for a degenerate advection-diffusion problem. *Interfaces and Free Boundaries*, 19, 371-391.
76. Ekal, G. B., Chew, J. V. L., Ghazali, K., Sulaiman, J., Xu, M. Half-sweep similarity Newton-MKSOR iterative method to solve the forced porous medium equation. *Applied Numerical Heat Transfer and Fluid Mechanics*, 38, 6382.
77. Helwani, N., Moufawad, S. M., Nassif, N. A two-stage learning PINN approach for solving the inverse problem of the 1D porous medium equation. Preprint (arXiv), , .
78. Dalibard, A.-L., Lopez-Ruiz, G., Perrin, C. Traveling waves for the porous medium equation in the incompressible limit: asymptotic behavior and nonlinear stability. *Indiana University Mathematics Journal*, 73, 973-1016.
79. Penington, S. The spreading speed of solutions of the non-local Fisher-KPP equation. *Journal of Functional Analysis*, 275, 3259-3302.
80. Hamel, F., Ryzhik, L. On the nonlocal Fisher-KPP equation: steady states, spreading speed and global bounds. *Nonlinearity*, 27, 2735-2753.
81. Calvo J, Campos J, Caselles, V., Sanchez, O., Soler, J. Flux-saturated porous media equations and applications. *ESAIM: Proceedings and Surveys*, 50, 1-19.
82. Berestycki, H., Nadin, G., Perthame, B., Ryzhik, L. The non-local Fisher-KPP equation: Travelling waves and steady states. *Nonlinearity*, 22, 2813-2844.
83. Wang, ZC., Li, W.-T., Ruan, S. Existence and stability of traveling wave fronts in reaction advection diffusion equations with nonlocal delay. *Journal of Differential Equations*, 233, 210-238.
84. Li, P., Yuan, R. Spreading dynamics of stochastic Fisher-KPP equations perturbed by mean-reverting processes. *Nonlinearity*, 39, 025013.
85. Li, P., Zhang, X., Yuan, R. Spreading dynamics for stochastic Fisher-KPP equation with shifting environment. *Mathematische Annalen*, 39.
86. Oelschläger, K. Large systems of interacting particles and the porous medium equation. *Journal of Differential Equations*, 88, 294-346.
87. Chayes, J. T., Panferov, V. Long-range interactions in competition models: the slow diffusion limit. *Nonlinear Analysis: Real World Applications*, 11, 4323-4340.
88. Berestycki, H., Nirenberg, L. Travelling fronts in cylinders. *Annales de l'Institut Henri Poincaré, Analyse Non Lineaire*, 9, 497-524.
89. Mei, M. Asymptotic stability of traveling waves for quasilinear parabolic systems. *Journal of Differential Equations*, 227, 355-388.
90. Ji S, Wang ZA, Xu T, Yin J. A reducing mechanism on wave speed for chemotaxis systems with degenerate diffusion. *Calc Var Partial Differential Equations*. 2021;60:178.