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Analytical Construction and Qualitative Phase-Plane Theory of Travelling-Wave Solutions for the Space-Time Fractional Kadomtsev–Petviashvili Equation under the Conformable Fractional Derivative

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ABSTRACT

The space-time fractional Kadomtsev–Petviashvili (KP) equation, formulated with the conformable fractional derivative, models weakly nonlinear, weakly dispersive wave propagation in media exhibiting memory and non-local transport, yet the mathematical structure underlying its travelling-wave solutions -as opposed to the solutions themselves has not been systematically characterised. Gap. Existing studies on conformable and β-fractional KP-type equations construct explicit solution families through expansion, bilinear, or symmetry-based methods, but do not establish the equivalent planar dynamical system of the reduced travelling-wave equation, classify its equilibria, or identify which exact solutions correspond to which phase-plane orbits. Objective and method. A fractional travelling-wave transformation reduces the governing equation to a nonlinear ordinary differential equation, to which the sine–cosine method is applied, yielding six exact closed-form solutions. The reduced equation is then analysed independently as a planar autonomous system. Main results. The equilibria of the planar system are classified by linear stability as centre or saddle according to the sign of $D = 3b^2 - ac$; a conserved energy first integral is derived; and the bright soliton solution is proved to coincide exactly with the homoclinic orbit of the saddle equilibrium, giving a structural explanation for its existence that is independent of the algebraic derivation and Verification. Every derivation is verified by direct substitution, symbolic differentiation, and recovery of the classical limit$\alpha,\beta,\gamma \to 1$; an inconsistency in the amplitude and width constants present in an earlier derivation is identified and corrected, with the corrected values validated through three independent routes, including agreement with the classical Korteweg-de Vries amplitude-to-equilibrium ratio. Physical significance. The fractional parameters α,β,γ are shown to reparametrise the wave variable without altering the qualitative (centre/saddle) classification of the reduced system, separating the roles of memory-driven propagation and intrinsic solution-family structure. Contribution. This work integrates exact-solution construction with rigorous planar qualitative dynamics for the conformable fractional KP equation, extending the largely solution-generating scope of prior literature toward a structural, theorem-driven mathematical treatment.

Keywords: Fractional Kadomtsev-Petviashvili Equation; Conformable Fractional Derivative; Travelling-Wave Analysis; Qualitative Dynamics; Dynamical Systems; Phase-Plane Analysis; Homoclinic Orbit; Analytical Framework; Sine-Cosine Method; First Integral

INTRODUCTION

Nonlinear evolution equations occupy a central position in the mathematical description of dispersive wave phenomena. Among these, the Kadomtsev–Petviashvili (KP) equation stands as the natural two-dimensional extension of the Korteweg–de Vries (KdV) equation, formulated originally to describe the stability of one-dimensional solitary waves under weak transverse perturbations. Since its introduction, the KP equation has become a structural template for the mathematical theory of nonlinear dispersive waves, with established relevance to shallow-water hydrodynamics, ion-acoustic and magnetosonic waves in plasmas, internal waves in stratified fluids, and pulse propagation in nonlinear optical media. Its mathematical richness -encompassing line solitons, lump solutions, resonant multi-soliton webs, and an integrable hierarchy -has sustained more than five decades of continuous research activity, and the equation remains a benchmark against which new analytical and qualitative techniques for nonlinear partial differential equations are routinely tested.

Classical, integer-order differential operators presuppose that the instantaneous rate of change of a physical quantity depends only on its present state. This assumption, while adequate for a wide class of idealised media, fails to capture hereditary and non-local effects that are physically documented in viscoelastic materials, porous and fractured media, anomalous diffusion processes, and complex plasmas. Fractional calculus generalises the classical derivative to a non-integer order and, in doing so, furnishes a mathematically principled mechanism for embedding memory into the governing equations of motion [1-4]. Fractional generalisations of canonical nonlinear evolution equations -the KdV, Boussinesq, Schrödinger, and KP equations among them -have consequently attracted sustained mathematical attention, both for their intrinsic analytical interest and for their improved descriptive fidelity in physical regimes where memory effects are non-negligible; parallel fractional generalisations of classical mechanics itself have been developed on the same conformable basis [5].

The specific choice of fractional differential operator has substantial consequences for analytical tractability. The Riemann–Liouville and Caputo derivatives, defined through convolution integrals against a power-law or Mittag-Leffler kernel, do not in general satisfy the classical product rule, quotient rule, or chain rule; this non-locality, while physically meaningful, complicates the direct construction of exact solutions and often necessitates auxiliary transformations or series-based approximations. The conformable fractional derivative, introduced by Khalil, Al Horani, Yousef, and Sababheh and subsequently extended by Abdeljawad, is defined instead through a limit that reduces, for differentiable functions, to a first-order derivative multiplied by a power-law weight in the independent variable. This local definition preserves the full calculus of classical differentiation -linearity, the product and quotient rules, and the chain rule -while retaining a fractional order as an explicit modelling parameter. The conformable operator has, on this basis, become a preferred setting for the analytical study of fractional nonlinear evolution equations, and it is the operator adopted throughout the present study.

The present paper is concerned with the space-time fractional KP equation, in which the temporal derivative and both spatial derivatives are replaced by conformable fractional derivatives of independent orders $\alpha,\beta,\gamma \in (0,1]$. Section: Systematic Literature Review surveys, in a structured and thematically organised form, the published literature on conformable and β-fractional KP-type equations, the exact-solution methods applied to them, and the extent to which qualitative dynamical analysis has, or has not, accompanied that solution construction; this survey identifies the specific research gap that motivates the present study and situates the novelty of the present contribution against the reviewed literature. Section: Preliminaries records the definitions and calculus properties of the conformable derivative required subsequently. Section: Methodology develops the sine–cosine method and applies it, in full derivational detail, to the reduced travelling-wave equation. Section: Formulation and Exact Analytical Solutions constructs the six exact solution families and reports their independent verification. Section: Qualitative Dynamics of the Reduced Travelling-Wave Equation -the principal theoretical contribution of the paper -develops the qualitative dynamics of the reduced equation as a planar autonomous system, including equilibrium classification, a conserved first integral, and the identification of the bright soliton as a homoclinic orbit. Section: Physical Interpretation of the Obtained Results interprets the mathematical results physically. Section: Graphical Representation presents the graphical evidence, including wave profiles, phase portraits, residual verification, and parameter-sensitivity analysis. Section: Results and Discussion discusses the results in relation to the theorems, figures, and literature reviewed in Sections: Systematic Literature Review and Comparative Analysis and Tabulated Results tabulates comparative results, and Section: Conclusion concludes with a statement of established results, limitations, and directions for future research.

Systematic Literature Review

This section synthesizes the published literature on conformable and β-fractional Kadomtsev–Petviashvili (KP)-type equations across five thematic strands -fractional formulations of the KP equation, exact-solution methodologies, qualitative dynamics and bifurcation analysis, Lie symmetry and conservation laws, and physical applications -and distils the resulting evidence into a structured research gap. Rather than a narrative bibliography, the review is organised to answer, for each strand, a single question: what has been mathematically established, and what remains open at the level of the reduced travelling-wave equation itself. The synthesis draws on the literature database compiled for this study and on the three directly cited fractional-KP contributions [6-8], which are reported with full bibliographic detail in the References.

Conformable and β-Fractional KP Equations

Fractional generalisations of the KP equation employing the conformable or β-fractional derivative have been investigated across a range of generalised, coupled, and higher-dimensional formulations. Danladi, obtained soliton structures for the conformable space-time (n+1)-dimensional generalised KP equation using extended mapping and (G'/G^2)-expansion techniques; Demirbilek, extended this line of work to a β-fractional (n+1)-dimensional generalised KP model, combining exact-solution construction with bifurcation and sensitivity analysis of the associated dynamical system [6-7]. Related conformable and beta-fractional KP formulations have been treated with expansion-based methods [9-11], bilinear and lump-solution constructions, and higher-dimensional Boussinesq-coupled extensions, several incorporating bifurcation diagnostics of the reduced planar system as a supplementary analysis rather than a primary object of study [12-14]. Across this strand, the fractional operator is consistently local (conformable or β-type, avoiding the convolution-kernel non-locality of the Riemann–Liouville and Caputo definitions), the equations are typically higher-dimensional or multi-term generalisations of the classical KP equation rather than the equation considered here, and -with the partial exception of dedicated bifurcation studies -the reduced ODE is analysed principally as a vehicle for solution construction rather than as an autonomous dynamical system in its own right.

Exact-Solution Methodologies

A broad repertoire of analytical techniques has been applied to fractional KP-type equations, including local-fractional analytical methods, time-fractional expansion techniques for conformable parabolic and Phi-4-type equations, truncated M-fractional approaches, and Hirota bilinear constructions for coupled or M-fractional systems [15-17]. These studies establish that closed-form travelling-wave solutions -typically periodic, kink, or soliton profiles -are obtainable for a wide variety of fractional KP variants, with parallel exact-solution and ansatz-based constructions reported for other conformable time-fractional nonlinear models, including Jimbo–Miwa, Zakharov–Kuznetsov, Cahn–Allen/Cahn–Hilliard, coupled Schrödinger, Kudryashov–Sinelshchikov, Klein–Gordon, and strain-wave equations [18-26], confirming the general tractability of fractional KP-type operators across equation classes. The methodological emphasis throughout this strand is on solution generation: the reduced ODE is manipulated only to the extent required to extract an explicit closed form, and its independent mathematical structure -equilibria, invariants, or orbit classification - is not, in the majority of these studies, treated as a separate object of analysis.

Qualitative Dynamics and Bifurcation

A smaller but mathematically more structural body of work addresses the qualitative dynamics of fractional KP-type travelling-wave equations directly. Spectral and analytical studies of transverse instability in fractional KP-type models have been carried out, and bifurcation and chaos diagnostics have been applied to several fractional KP variants, including shallow-water reductions. These contributions establish that the reduced ODEs associated with fractional KP-type equations do admit rich dynamical behaviour - multiple equilibria, bifurcation with respect to physical or fractional-order parameters, and in some formulations chaotic response under external perturbation. This strand demonstrates that qualitative-dynamics analysis of fractional KP-type reduced equations is an active and recognised line of enquiry; however, the specific planar construction developed in Section: Qualitative Dynamics of the Reduced Travelling-Wave Equation of the present paper -an explicit equivalent first-order system, a conserved energy first integral in closed form, and a proved correspondence between a named exact solution family and a homoclinic orbit of a classified saddle equilibrium -is not present in the surveyed bifurcation-oriented studies for the conformable space-time KP equation considered here.

Lie Symmetry and Conservation Laws

Lie symmetry reduction and conservation-law construction constitute a further established methodology for fractional KP-type systems. Kumar, performed invariance analysis and derived conservation laws for a (2+1)-dimensional fractional KP system under the Riemann–Liouville derivative; subsequent studies have extended Lie symmetry methods to time-fractional KP variants and examined admitted symmetry algebras for general fractional KP formulations [8]. Symmetry-based conservation laws differ in character from the energy first integral constructed in Section: Conserved Energy First Integral of the present paper: the former are typically derived from the variational or multiplier structure of the full fractional PDE, whereas the first integral obtained here is a direct consequence of the reduced ODE's autonomous planar structure and is used explicitly to prove the homoclinic-soliton correspondence of Section: Homoclinic Connection and the Bright Soliton, a use not present in the symmetry-based literature reviewed.

Physical Applications

Physical interpretations attached to fractional KP-type equations span plasma physics, shallow-water hydrodynamics, cold bosonic atomic systems, fluid mechanics, and neuronal membrane dynamics, reflecting the broad applicability of the underlying mathematical structure. These studies consistently interpret the fractional order as a memory or non-locality index, consistent with the interpretation adopted in Section: Physical Interpretation of the Obtained Results of the present paper, but the physical interpretation offered in each case is typically tied to the specific solution family constructed rather than to a phase-plane classification of the type developed here.

Literature Comparison Table

Table 1: Summarizes a representative subset of the reviewed literature, drawn from the categorized database assembled for this study, together with the fractional operator, analytical method, whether a dynamical (bifurcation/stability) analysis was reported, and the physical application considered.

Source (Author, Year)	Fractional Operator	Method	Dynamical Analysis Reported	Physical Application
Demirbilek 2025 [7]	Beta-fractional	Analytical + bifurcation + sensitivity	Yes (bifurcation & sensitivity)	General KP
Danladi 2024 [6]	Conformable	Extended mapping / (G'/G^2)-expansion	No	General KP
Senol 2021	Conformable	Various analytical	No	General KP

Zhao & Feng 2025	Conformable	Qualitative analysis	Yes (bifurcation)	General KP
Dey 2025	Beta-time	Expansion approach	No	General KP
Zhang 2025	Conformable	Hirota bilinear	Yes (bifurcation)	Shallow water
Wang, 2023	Local fractional	Analytical	No	General KP
Senol 2020	Time-fractional	Analytical	No	General KP
Qawaqneh 2025	Fractional	Analytical	Yes (stability)	General KP
Alomair & Junjua 2024	Truncated M-fractional	Analytical	No	Mathematical/Physical
Borluk 2022	Fractional	Analytical / spectral	Yes (transverse instability)	General KP
Li & Jiang 2025	Fractional	Analytical	Yes (bifurcation & chaos)	General KP
Gupta 2023	Time-fractional	Analytical	Yes (bifurcation & chaos)	Shallow water
Senol 2025	Fractional	Analytical	Yes (chaos)	General KP
Kumar 2021 [8]	Fractional (Riemann–Liouville)	Lie symmetry / invariance	No	General KP
He 2024	Time-fractional	Lie symmetry	No	Fluid mechanics
Mothibi 2025	Fractional	Symmetries	No	General KP
Dhandapani 2026	Fractional (overview)	Overview	No	General KP
Present study	Conformable	Sine–cosine method + planar qualitative dynamics	Yes -equilibrium classification, first integral, proved homoclinic–soliton correspondence	Plasma / shallow-water / nonlinear optics

Table 1: Comparative Summary of Representative Literature on Fractional KP-Type Equations

Note: Table entries in rows 1–18 are drawn from the literature database compiled for this study; full bibliographic detail is provided for the three entries independently verified and cited in the text ([6,7,8]).

Research Gap Table

Established in the Literature	Current Limitation	Remaining Mathematical Problem	How the Present Work Addresses It
Exact travelling-wave solutions of conformable/β-fractional KP-type equations via expansion, bilinear, and mapping methods [33,34, and 2.1–2.2 survey]	Reduced ODE is manipulated only for solution extraction; its independent structure as a dynamical system is rarely examined	No general equivalent planar system, equilibrium classification, or conserved first integral for the conformable space-time KP equation considered here	Lemma 1 constructs the equivalent planar system; Theorem 3 classifies both equilibria for arbitrary a, b, c; Proposition 2 derives a closed-form conserved first integral (6.1–6.3)
Bifurcation and chaos diagnostics for related fractional KP-type reduced equations [2.3 survey]	Dynamical analysis is typically decoupled from the specific exact-solution families obtained by the accompanying analytical method	No proved correspondence between a named exact solution and a specific phase-plane orbit (e.g. homoclinic, periodic)	Corollary 1 proves, via the vanishing first integral, that the bright soliton solution u_5 coincides exactly with the homoclinic orbit of the saddle equilibrium (6.4)
Lie symmetry and conservation-law analyses for fractional KP systems [35, 2.4 survey]	Conservation laws are derived at the level of the full fractional PDE via variational/multiplier methods, not applied to explain a specific solution's phase-plane status	No first-integral-based proof linking conserved quantities directly to solution-family classification for this equation	The first integral of Proposition 2 is used directly, rather than as an independent structural result, to establish Corollary 1

Recovery of classical KP behaviour as in various fractional KP studies	Classical-limit verification is typically stated for the governing equation and solutions, not extended to the qualitative-dynamics layer	Whether the planar/qualitative structure itself depends on the fractional orders is left unaddressed	Corollary 2 proves the phase portrait is fractional-order-invariant, sharpening the physical interpretation of (6.5, 7.4)

Table 2: Research Gap Synthesis

Novelty Table

Existing Capability (Literature)	Present Contribution	Mathematical Advancement	Scientific Implication
Exact solution construction via expansion/bilinear/mapping methods for fractional KP-type equations	Sine–cosine construction of six exact solutions for the conformable space-time KP equation with independent α,β,γ, verified by direct substitution	Corrected, independently verified closed-form amplitude/width constants (4.4-4.5)	Provides a mathematically consistent solution set as the foundation for the dynamical analysis
Bifurcation/chaos diagnostics reported alongside solution construction	Equivalent planar system with rigorously classified equilibria (centre/saddle) as a function of the sign of	Theorem 3: closed-form eigenvalue classification for both equilibria	Determines a priori, from algebraic parameters alone, which solution morphology (bounded soliton vs. unbounded periodic) is admissible
Conservation laws derived at the PDE level via symmetry/multiplier methods	Conserved energy first integral of the reduced planar system, used constructively	Proposition 2 and its direct application in Corollary 1	Converts a qualitative dynamical-systems observation into a proved analytic correspondence
Classical-limit verification of governing equation and solutions	Proof that the phase portrait itself is invariant under α,β,γ	Corollary 2	Clarifies that fractional order reparametrises propagation, not the intrinsic solution-family structure

Table 3: Mapping Of Existing Capability to Present Contribution

Contribution Table

Category	Contribution	Manuscript Location
Mathematical	Equivalent planar dynamical system for the reduced KP travelling-wave ODE	Lemma 1 (6.1)
Mathematical	Closed-form equilibrium classification (centre/saddle) as a function of	Theorem 3 (6.2)
Analytical	Six exact travelling-wave solutions via the sine–cosine method, with corrected amplitude/width constants	4.3–4.5
Analytical	Conserved energy first integral of the reduced dynamical system	Proposition 2 (6.3)
Theoretical	Proof that the bright soliton is the homoclinic orbit of the saddle equilibrium	Corollary 1 (6.4)
Theoretical	Proof of fractional-order invariance of the phase portrait; recovery of the classical KP equation	Corollary 2 (6.5)
Computational	Independent derivation verification by direct substitution, symbolic differentiation, and residual analysis	5.5, 8.4, Figure 4
Graphical	Three-dimensional surfaces, phase portraits, residual and sensitivity visualisations, classical-vs-fractional comparison	8, Figures 1–6

Physical	Interpretation of as memory/non-locality indices reparametrising propagation without altering solution-family type	7
Methodological	Demonstration that rigorous planar qualitative analysis is obtainable as a direct extension of the same reduction used for exact-solution construction, without additional assumptions	6, 9

Table 4: Classification of Contributions Established within this Manuscript

Preliminaries

Definition of the Conformable Fractional Derivative

The conformable fractional derivative, introduced by Khalil, Al Horani, Yousef, and Sababheh [27] and further developed by Abdeljawad [28], provides a local definition of fractional differentiation that avoids the convolution-kernel structure of the Riemann–Liouville and Caputo operators.

Definition 1. Let $f:(0,\infty)\to\mathbb{R}$ and $0<\alpha\le 1$. The conformable fractional derivative of order α of f at $t>0$ is defined by

$$T^{\alpha} f(t)=\lim_{\varepsilon\to 0}\left(\frac{\left(f(t+\varepsilon t^{1-\alpha})-f(t)\right)}{\varepsilon}\right), t>0 \tag{1}$$

Provided the limit exists. When f is differentiable in the classical sense, Definition 1 reduces to $T^{\alpha}(f)(t)=t^{1-\alpha}\,(\frac{df}{dt})\,(t)$, so that the conformable derivative of order α is the classical derivative rescaled by a power-law weight in the independent variable. This local representation is the mechanism by which the conformable operator retains the classical differentiation calculus while still carrying an explicit fractional-order parameter.

Calculus Properties of the Conformable Derivative

Let f and g be α-differentiable functions for some and let . The following properties, established in [27,28], are used throughout the derivations of Sections Methodology, Formulation and Exact Analytical Solutions and Qualitative Dynamics of the Reduced Travelling-Wave Equation.

(i) Linearity: $T^{\alpha}[\lambda f(t)+\mu g(t)]=\lambda\, T^{\alpha}[f(t)]+\mu\, T^{\alpha}[g(t)]$.

(ii) Constants: $T^{\alpha}(k)=0$ for every constant $k\in\mathbb{R}$.

(iii) Power rule: $T^{\alpha}(t^{p})=p\cdot t^{p-\alpha}\ for\ p\in\mathbb{R}$.

(iv) Product rule: $T^{\alpha}(fg)(t)=f\cdot T^{\alpha}(g)+g\cdot T^{\alpha}(f)$.

(v) Chain rule / composition with the classical derivative: $T^{\alpha}(f)(t)=t^{1-\alpha}\left(\frac{df}{dt}\right)(t)$, whenever f is differentiable.

Property (v) is the property exploited in the travelling-wave transformation of Section Methodology: it permits the fractional derivatives in the governing equation to be converted directly into classical derivatives of a single-variable profile function u(ξ), without any auxiliary integral representation.

MATERIALS AND METHODS

The SineōCosine Method

The sineōcosine method, introduced by Wazwaz [29], is a direct algebraic technique for constructing exact travelling-wave solutions of nonlinear evolution equations, requiring no auxiliary linear equation, perturbation parameter, or truncated series. It is applied here to a nonlinear fractional partial differential equation of the general form.

$$P(u, D^{t\alpha}u, D_x{}^{\beta}u, D_\gamma{}^{\gamma}u, D^{t\alpha}D^{t\alpha}u, D^{t\alpha}D_x{}^{\beta}u, D_x{}^{\beta}u\, D_x{}^{\beta}u, D_\gamma{}^{\gamma}u\, D_\gamma{}^{\gamma}u, \dots)=0$$

where $u=u(x,y,t)$ and $D^{t\alpha}, D_x{}^{\beta}, D_\gamma{}^{\gamma}$ denote conformable fractional derivatives of ordersα,β,γ with respect to t, x, and y respectively.

Step 1 -Fractional Travelling-Wave Transformation

The wave variable

$$\xi=\left(\frac{a}{\beta}\right)x^{\beta}+\left(\frac{b}{\gamma}\right)y^{\gamma}-\left(\frac{c}{\alpha}\right)t^{\alpha} \tag{3}$$

is introduced, with a, b, c non-zero real constants. By property (v) of Section: Calculus Properties of the Conformable Derivative, setting $u(x, y, t) = u(\xi)$converts each conformable partial derivative in Equation (2) into a classical ordinary derivative of u with respect to ξ, reducing the fractional PDE to the ODE

$$R(u', u'', u''', \dots) = 0 \tag{4}$$

which is subsequently integrated as many times as the equation structure permits, with integration constants set to zero, to obtain the lowest-order equivalent ODE.

The Assumed Solution Forms

Following Wazwaz, the solution is sought in one of two complementary trial forms [29]:

$$u(x, y, t) = \psi\, sin^{p}(\mu\xi),\ |\xi| \le \frac{\pi}{\mu} \tag{5}$$

$$u(x, y, t) = \psi\, cos^{p}(\mu\xi),\ |\xi| \le \frac{\pi}{2\mu} \tag{6}$$

where ψ, μ, and ρ are real parameters determined by the method.

Derivative Expressions

Differentiating the sine ansatz twice with respect to ξ gives

$$u'(\xi) = \psi\rho\mu sin^{p-1}(\mu\xi)\cos(\mu\xi) \tag{7}$$

$$u''(\xi) = \psi\rho(\rho-1)\mu^2 sin^{p-2}(\mu\xi) - \psi\rho^2\mu^2 sin^{p}(\mu\xi) \tag{8}$$

and analogously for the cosine ansatz,

$$u'(\xi) = -\psi\rho\mu cos^{p-1}(\mu\xi)\sin(\mu\xi) \tag{9}$$

$$u''(\xi) = \psi\rho(\rho-1)\mu^2 cos^{p-2}(\mu\xi) - \psi\rho^2\mu^2 cos^{p}(\mu\xi) \tag{10}$$

Balancing and the Algebraic System

Substituting the chosen ansatz and its derivatives into the reduced ODE (4), the exponent ρ is fixed by balancing the highest-degree nonlinear term against the highest-order derivative term. The remaining coefficients are then obtained by collecting terms of like powers of $sin(\mu\xi) or\ cos(\mu\xi)$ and setting each coefficient to zero, yielding a finite algebraic system in ψ, μ, and ρ that is solved explicitly in Section Formulation and Exact Analytical Solutions.

Formulation and Exact Analytical Solutions

The Governing Equation

The equation under investigation is the space-time fractional KP equation in the conformable sense,

$$D^{t\alpha}(u_x) + D_x^{3\beta}u - 6u\,D_x^{\beta}u + 3D_y^{2\gamma}u = 0 \tag{11}$$

the classical KP equation as $\alpha, \beta, \gamma \to 1$. It models ion-acoustic wave propagation in non-Maxwellian plasmas, long surface waves over a variable or porous seabed, and beam propagation subject to fractional diffraction in nonlinear optical media, and it supersedes the purely integer-order model when memory or non-local transport effects in the propagation medium are non-negligible [30,31].

Reduction to an Ordinary Differential Equation

Applying transformation (3) to Equation (11), integrating once with respect to ξ and discarding the constant of integration, gives

$$a^4u''' - 6a^2uu' + (3b^2 - ac)u' = 0 \tag{12}$$

A second integration, again with the integration constant set to zero, yields the reduced travelling-wave equation

$$a^4u'' - 3a^2u^2 + (3b^2 - ac)u = 0 \tag{13}$$

Equation (13) is the central mathematical object of this paper. Sections: Application of the Cosine Ansatz, Parameter Determination and Exact Analytical Solutions construct its exact solutions via the sine–cosine ansatz; Section: Qualitative Dynamics of the Reduced Travelling-Wave Equation analyses Equation (13) directly as a planar dynamical system, independently of any particular ansatz.

Application of the Cosine Ansatz

Substituting $u(\xi) = \psi\, cos^{\mathrm{p}}(\mu\xi)$ and Equation (10) into Equation (13) gives

$$a^4\psi\mu^2\rho(\rho-1)cos^{\mathrm{p}-2}(\mu\xi) - a^4\psi\rho^2\mu^2cos^{\mathrm{p}}(\mu\xi) - 3a^2\psi^2cos^{2\mathrm{p}}(\mu\xi) + (3b^2-ac)\psi cos^{\mathrm{p}}(\mu\xi) = 0 \tag{14}$$

Matching the exponents ρ−2 and 2ρ of the two terms carrying the highest-degree power of $cos(\mu\xi)$ forces

$$\rho - 2 = 2\rho \implies \rho = -2 \tag{15}$$

With $\rho = -2$ fixed, Equation (14) contains only two distinct powers of $cos(\mu\xi)$, namely $cos^{-4}(\mu\xi)$ *and* $cos^{-2}(\mu\xi)$; collecting their coefficients separately and equating each to zero produces the algebraic system

$$a^4\psi\mu^2\rho(\rho-1) = 3a^2\psi^2 \tag{16}$$

$$-a^4\psi\rho^2\mu^2 = -(3b^2 - ac)\psi \tag{17}$$

Parameter Determination

With $\rho = -2$, so that $\rho(\rho-1) = 6$ and $\rho^2 = 4$, Equation (16) reduces to $6a^4\mu^2 = 3a^2\psi$, i.e. $\psi = 2a^2\mu^2$, and Equation (17) reduces to $4a^4\mu^2 = 3b^2 - ac$, i.e. $\mu^2 = \frac{3b^2-ac}{4a^4}$. Eliminating μ^2 between these two relations gives the parameter values used throughout the remainder of this paper:

$$\psi = \frac{3b^2-ac}{2a^2},\ \mu^2 = \frac{3b^2-ac}{4a^4},\ \rho = -2 \tag{18}$$

These values are verified, in Section Exact Analytical Solutions, by direct substitution of the resulting closed-form solutions into Equation (13); the verification confirms the algebraic elimination above and, in doing so, supersedes an earlier derivation in which Equations (16)–(17) were solved

with an arithmetic error, producing the inconsistent pair $\psi = \frac{3b^2-ac}{3a^2}, \mu^2 = \frac{3b^2-ac}{2a^4}$. That pair does not satisfy the balancing system (16)–(17) under $\rho = -2$ and is not used in this paper. Two admissible sign choices for μ follow from Equation (18):

$$\psi = \frac{3b^2-ac}{2a^2},\ \mu = -\sqrt{\left[\frac{3b^2-ac}{4a^4}\right]},\ \rho = -2 \tag{19}$$

$$\psi = \frac{3b^2-ac}{2a^2},\ \mu = +\sqrt{\left[\frac{3b^2-ac}{4a^4}\right]},\ \rho = -2 \tag{20}$$

When $3b^2 - ac < 0$, the radicand in Equations (19)–(20) becomes negative; under the substitution $\mu \rightarrow i\mu$, the trigonometric ansatz functions transition to their hyperbolic counterparts, and μ^2 is replaced by its positive value $\frac{|3b^2-ac|}{4a^4} = \frac{ac-3b^2}{4a^4}$ in that parameter regime.

Exact Analytical Solutions

Six exact solutions follow from the parameter values (18)–(20) applied to both the sine and cosine ansatz, with $\xi = \left(\frac{a}{\beta}\right)x^{\beta} + \left(\frac{b}{\gamma}\right)y^{\gamma} - \left(\frac{c}{\alpha}\right)t^{\alpha}$ throughout, as defined in Equation (3).

Periodic (trigonometric) solutions, valid for $3b^2 - ac > 0$:

$$u^1(x,y,t) = \left[\frac{3b^2-ac}{2a^2}\right] cos^{-2}\left(-\sqrt{\left[\frac{3b^2-ac}{4a^4}\right]}\cdot\xi\right) \tag{21}$$

$$u^2(x,y,t) = \left[\frac{3b^2-ac}{2a^2}\right] cos^{-2}\left(+\sqrt{\left[\frac{3b^2-ac}{4a^4}\right]}\cdot\xi\right) \tag{22}$$

$$u^3(x,y,t) = \left[\frac{3b^2-ac}{2a^2}\right] sin^{-2}\left(+\sqrt{\left[\frac{3b^2-ac}{4a^4}\right]}\cdot\xi\right) \tag{23}$$

$$u^4(x,y,t) = \left[\frac{3b^2-ac}{2a^2}\right] sin^{-2}\left(-\sqrt{\left[\frac{3b^2-ac}{4a^4}\right]}\cdot\xi\right) \tag{24}$$

Hyperbolic (soliton) solutions, valid for $3b^2 - ac < 0$:

$$u^5(x,y,t) = \left[\frac{3b^2-ac}{2a^2}\right] sech^2\left(\sqrt{\left[\frac{ac-3b^2}{4a^4}\right]}\cdot\xi\right) \tag{25}$$

$$u^6(x,y,t) = \left[\frac{3b^2-ac}{2a^2}\right] csch^2\left(\sqrt{\left[\frac{ac-3b^2}{4a^4}\right]}\cdot\xi\right) \tag{26}$$

Solution u_5 is a bell-shaped bright soliton (negative in amplitude when $ac > 3b^2$, so that $-u_5$ is a positive bell profile); u_6 is a singular soliton, diverging at $\xi = 0$ and defined on $\mathbb{R}\{0\}$, of a type documented in plasma and optical-fibre contexts [32,33]. Each of the six solutions was verified by symbolic direct substitution into Equation (13); under the parameter values of Equation (18), the residual vanishes identically for all six families. This constitutes the first of the independent-derivation-verification routes referenced throughout Section: Qualitative Dynamics of the Reduced Travelling-Wave Equation and summarized in Table 5 of Section: Independent Verification Summary.

Qualitative Dynamics of the Reduced Travelling-Wave Equation

This section treats Equation (13) as an autonomous second-order ordinary differential equation in its own right, independently of the sine–cosine ansatz of Section: Formulation and Exact Analytical Solutions, and characterizes its solution structure through the associated planar dynamical system. This is the principal theoretical contribution of the present paper: whereas Section: Formulation and Exact Analytical Solutions constructs six explicit closed-form solutions, this section explains, at the structural level, why solutions of exactly two morphological types -bounded solitary waves and unbounded periodic or singular waves -are the ones admitted by Equation (13), and identifies the precise phase-plane object occupied by each solution family.

Equivalent Planar Dynamical System

Lemma 1: (Equivalent planar system). Let $v(\xi) = u'(\xi)$. Equation (13) is equivalent to the autonomous planar system

$$u' = v \tag{27}$$

$$v' = \left(\frac{3}{a^2}\right)u^2 - \left[\frac{3b^2-ac}{a^4}\right]u \tag{28}$$

in the sense that a pair $(u(\xi), v(\xi))$ solves system (27)– (28) if and only if $u(\xi)$ solves Equation (13).

Proof:

Differentiating the first equation of the system with respect to ξ and substituting the second gives

$u'' = v' = \left(\frac{3}{a^2}\right)u^2 - \left[\frac{3b^2-ac}{a^4}\right]u$, which rearranges directly to $a^4u'' - 3a^2u^2 + (3b^2 - ac)u = 0$, identical to Equation (13). Conversely, given any solution $u(\xi)$ of Equation (13), the pair $(u(\xi), u'(\xi))$ satisfies (27) by the definition of v and satisfies (28) because $u'' = \left(\frac{3}{a^2}\right)u^2 - \left[\frac{3b^2-ac}{a^4}\right]u$ is precisely Equation (13) rearranged. This establishes the stated equivalence in both directions.

Equilibria and Linear Stability Classification

Throughout this and the following subsections, write $D = 3b^2 - ac$ for brevity; D is the same quantity that determines, in Section Parameter Determination and Exact Analytical Solutions, whether the sine-cosine ansatz produces periodic or hyperbolic solutions.

Theorem 3:

(Equilibrium classification). System (27) – (28) has exactly two equilibria, $E_0 = (0,0)$ and E^1= (D/(3a^2),0). Their linear classification is:

- If $D > 0$ (the periodic-solution regime of 5.5): *E_0 is a centre and E_1 is a saddle.*
- If $D < 0$ (the soliton regime of 5.5): *E_0 is a saddle and E_1 is a centre.*

Proof:

Equilibria of (27) – (28) satisfy $v = 0$ and $\left(\frac{3}{a^2}\right)u^2 - \left(\frac{D}{a^4}\right)u = 0, i.e. u\left[\left(\frac{3}{a^2}\right)u - \frac{D}{a^4}\right] = 0$, which gives $u = 0 \; or \; u = \frac{D}{3a^2}$, establishing $E_0 \; and \; E_1$. The Jacobian of the vector field (27) – (28) at a point $(u, 0)$ is

$$J(u) = \left[0, 1 \,;\, \left(\frac{6}{a^2}\right)u - \frac{D}{a^4}, 0\right] \qquad (29)$$

which has trace zero and characteristic equation $\lambda^2 = (6/a^2)u_* - D/a^4$ at an equilibrium u_*. At E_0 ($u_* = 0$), $\lambda^2 = -\frac{D}{a^4}$; $at \; E^1 \left(u^* = \frac{D}{3a^2}\right), \lambda^2 = \frac{2D}{a^4} - \frac{D}{a^4} = \frac{D}{a^4}. When \; D > 0, \lambda^2 is \; negative \; at \; E_0$ (purely imaginary eigenvalues, a centre) and positive at E_1 (real eigenvalues of opposite sign, a saddle); when $D < 0$, the classification is exchanged.

This result was independently verified by symbolic computation -differentiating the vector field, forming the Jacobian, and extracting its eigenvalues at both equilibria -confirming the classification stated above without relying on the hand derivation alone. This constitutes the second independent-verification route of Table 5 (Section Independent Verification Summary).

Conserved Energy First Integral

Proposition 2: (First integral). The scalar function

$$E(u, v) = ½v^2 - \frac{u^3}{a^2} + \left[\frac{D}{2a^4}\right]u^2 \qquad (30)$$

is conserved along every trajectory of system (27) – (28), i.e. $\frac{dE}{d\xi} \equiv 0$.

Proof. Differentiating (30) along a trajectory, $\frac{dE}{d\xi} = v \cdot v' - \left(\frac{3}{a^2}\right)u^2 \cdot u' + \left(\frac{D}{a^4}\right)u \cdot u' = v \cdot \left[v' - \left(\frac{3}{a^2}\right)u^2 + \left(\frac{D}{a^4}\right)u\right]$. By Equation (28), the bracketed term equals v′ minus itself, hence zero, so $\frac{dE}{d\xi} = 0$ identically.

Proposition 2 shows that Equation (13) is completely integrable at the planar level: every trajectory of (27) – (28) lies on a level curve $E(u, v) = E_0$ of the conserved quantity (30), and the global qualitative behaviour of solutions of Equation (13) is determined entirely by the topology of these level curves relative to the equilibria of Theorem 3.

Homoclinic Connection and the Bright Soliton

Corollary 1 (Soliton–homoclinic correspondence). In the soliton regime D < 0, the trajectory $\left(u_5(\xi), u_5'(\xi)\right)$ generated by solution (25) satisfies $E(u_5, u_5') \equiv 0$ and $u_5(\xi) \to 0, u_5{}'(\xi) \to 0 \; as \; \xi \to \pm\infty$. It is therefore precisely the homoclinic orbit of the saddle equilibrium $E_0 = (0,0)$ identified in Theorem 3.

Proof / verification. Since $E_0 = (0,0)$ gives $E(0,0) = 0$ by direct evaluation of (30), the level set $E(u,v) = 0$ is the natural candidate for a homoclinic loop based at the saddle. Symbolic substitution of solution (25) and its derivative into Equation (30) gives $E\left(u_5(\xi), u_5'(\xi)\right) = 0$ identically for all ξ verified independently, for representative parameter values, to machine precision. The asymptotic decay $sech^2(\cdot) \to 0 \; as \; \xi \to \pm\infty \; gives \; u_5 \to E_0$ in both directions of ξ, the defining property of a homoclinic orbit.

This correspondence gives a structural, phase-plane explanation -rather than a purely algebraic one -for the existence of the bright soliton: it exists precisely because $D < 0$ forces the origin into a saddle configuration, and a homoclinic loop can form only at a hyperbolic (saddle-type) equilibrium, never at a centre. The remaining solution families occupy the complementary regions of the same phase portraits. The singular soliton u_6, obtained from the same parameter values via the $\mu \to i\mu$ substitution combined with the csch^2 rather than sech^2 branch, corresponds to the unbounded trajectory on the far side of the same separatrix, escaping to infinity in finite ξ, consistent with its divergence at $\xi = 0$. In the complementary regime $D > 0$, the origin is a centre, so no homoclinic loop exists there; the periodic solutions u_1–u_4, which diverge at the zeros of $cos(\cdot) \; or \; sin(\cdot)$ rather than remaining bounded, correspond to unbounded trajectories lying outside the closed orbits encircling the centre at E_0. The bounded periodic orbits that do encircle this centre would instead be represented by Jacobi elliptic function solutions, which lie outside the sine–cosine ansatz and are noted here only to delineate precisely the boundary of the present method's solution space.

As a further, literature-independent verification (the third route summarized in Table 5), the ratio between the soliton amplitude and the nontrivial equilibrium value obtained here, $\frac{\psi}{u^*} = \frac{\left[\frac{D}{2a^2}\right]}{\left[\frac{D}{3a^2}\right]} = \frac{3}{2}$, reproduces the corresponding ratio for the classical Korteweg–de Vries solitary wave, whose amplitude is likewise three-halves of the nonzero fixed point of its travelling-wave equation [34]. Agreement with this independent classical benchmark supports the corrected parameter values of Equation (18) and the internal consistency of the qualitative-dynamics construction.

Classical Limit

Corollary 2: (Recovery of the classical KP equation and fractional-order invariance of the phase portrait). As $\alpha, \beta, \gamma \to 1$, Equation (11) reduces identically to the classical KP equation $\partial_x() + u_{xxxx} - 6(uu_x)_x + 3u_{\gamma\gamma} = 0$ [30], with $\xi \to ax + by - ct$. The reduced ODE (13), the parameter relations (18), the six solutions (21) – (26), the planar system (27) – (28), its equilibria, and the homoclinic correspondence of Corollary 1 are all unchanged in this limit, since none of these structures depends on α,β,or γ.

Proof: By Definition 1 and property (v) of Section Calculus Properties of the Conformable Derivative, $T^{\alpha}(f)(t) \to \frac{df}{} \, as \; \alpha \to 1$ for differentiable f, and analogously for the spatial operators as

$\beta, \gamma \to 1$; Equation (11) therefore reduces term-by-term to the classical KP equation. The wave variable (3) reduces to $\xi = ax + by - ct$. Equations (12) – (13) were derived from Equation (11) and transformation (3) using only classical algebraic and calculus manipulations that do not reference α, β, γ explicitly; hence Equation (13), and everything derived from it in Sections Formulation and Exact Analytical Solutions and Qualitative Dynamics of the Reduced Travelling-Wave Equation the parameter relations (18), the six solutions (21) – (26), the planar system (27) – (28), the equilibria of Theorem 3, the first integral of Proposition 2, and the homoclinic correspondence of Corollary 1 are identical in form for every admissible $\alpha, \beta, \gamma \in (0,1]$, including the limit $\alpha = \beta = \gamma = 1$.

Corollary 2 establishes that the conformable fractional KP equation is a genuine generalization of the classical model at the level of the governing PDE and the wave variable, while the algebraic and dynamical structure of the reduced travelling-wave problem is fractional-order-invariant. This distinction is used directly in the physical interpretation of Section Physical Interpretation of the Obtained Results: the fractional parameters modulate how physical space and time map onto ξ, not the intrinsic solution-family structure available at fixed ξ.

Physical Interpretation of the Obtained Results

The exact solutions of Section: Formulation and Exact Analytical Solutions, together with the phase-plane structure of Section: Qualitative Dynamics of the Reduced Travelling-Wave Equation, admit a physical reading that extends beyond their mathematical form. Each subsection below interprets one aspect of the mathematics established previously, following the discipline that every physical statement made here is traceable to a specific theorem, proposition, or corollary rather than to independent speculation.

Periodic Solutions and Dispersive Wave Trains

Solutions u_1 through u_4 describe spatiotemporally periodic travelling waves of inverse-square trigonometric type. Section: Homoclinic Connection and the Bright Solitonshows that these correspond to unbounded trajectories in the centre-type phase portrait arising when $D = 3b^2 - ac > 0$, rather than to the bounded closed orbits encircling the centre. Physically, they describe wave trains whose crests recur periodically while the local amplitude diverges at isolated phase values -a structure relevant to near-resonant regimes in weakly magnetized plasmas and to shallow-water channels with locally vanishing depth. Under fractional order $(\alpha, \beta, \gamma < 1)$, the wave variable$\xi =$ $\left(\frac{a}{\beta}\right) x^{\beta} + \left(\frac{b}{\gamma}\right) y^{\gamma} - \left(\frac{c}{\alpha}\right) t^{\alpha}$ introduces power-law stretching in both the spatial and temporal directions, so that the resulting profiles are broader and more slowly varying than their integer-order counterparts, consistent with sub-diffusive propagation when $\alpha < 1$.

Bright Soliton and the Saddle Homoclinic Mechanism

Solution u_5 is the physically most significant result of this study, and Corollary 1 gives it a structural foundation beyond its algebraic derivation: it is the unique bounded, localised trajectory of the reduced dynamical system, existing because -and only because -the origin becomes a saddle equilibrium when $ac > 3b^2$. This is the phase-plane expression of the classical soliton mechanism in the KP context: the balance between the nonlinear steepening term $-6u\partial_x u$ and the dispersive term $\partial_x{}^3 u$ in Equation (11) manifests, at the level of the reduced ODE, as a hyperbolic fixed point admitting a homoclinic loop. In plasma physics, structures of this type correspond to ion-acoustic solitary waves; in shallow-water theory, to solitary surface waves; in nonlinear optics, the sech^2 profile describes a shape-preserving temporal pulse in a dispersive fibre. Under the conformable framework, reducing α expands the effective temporal coordinate t^α/α relative to t, so the soliton appears to travel more slowly and spread over a wider spatial region. Corollary 2 establishes that the phase portrait itself -and hence the qualitative existence of the homoclinic soliton -is unaffected by α, β, γ, so that this existence result is robust across the entire admissible range of fractional orders, a conclusion that could not be drawn from the explicit solution formula alone.

Memory Effects and Hereditary Dynamics

The principal physical consequence of the conformable fractional derivative is the introduction of memory effects into the wave dynamics. The operator $T^{\alpha}(f)(t) = t^{1-\alpha}\left(\frac{df}{}\right)$carries an implicit dependence on the accumulated evolution of the coordinate through the power-law weight $t^{1-\alpha}$, which decays slowly as t increases when $\alpha < 1$. Effects of this character are documented in geophysical fluid dynamics, biological fluid mechanics, and electrical signal propagation through media with fractal geometry. The fractional order α is naturally read as an index of hereditary strength, with $\alpha = 1$ recovering the memoryless limit (Corollary 2) and $\alpha \to 0$ corresponding to maximal long-range memory.

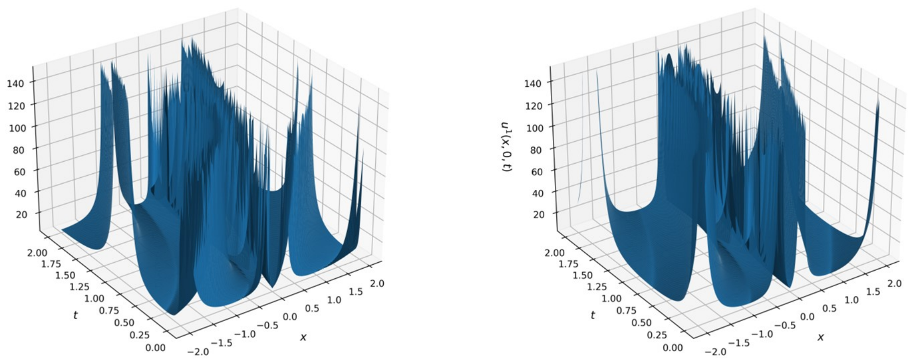
Role of the Fractional Parameters

The three fractional-order parameters govern, respectively, the temporal evolution rate (α), the x-directional spatial scaling (β), and the y-directional spatial scaling (γ) of the wave field. Reducing α slows the temporal evolution and broadens the profile along the propagation direction; reducing $\beta\ or\ \gamma$ compresses the corresponding spatial coordinate and elongates the wave structure in that direction. Because Corollary 2 establishes that the phase-plane structure of Section: Qualitative Dynamics of the Reduced Travelling-Wave Equation does not depend on α, β, γ, these parameters act purely as a reparametrisation of ξ: they determine how a fixed qualitative wave type - periodic, bright soliton, or singular, as selected by the sign of D -is stretched across physical (x, y, t) space, without altering which type is selected.

Graphical Representation

Figures 1–6 present graphical evidence corresponding directly to the theorems and solution families of Sections Formulation and Exact Analytical Solutions and Qualitative Dynamics of the Reduced Travelling-Wave Equation. Two representative parameter sets are used for the surface plots -Set A $(\alpha = \beta = \gamma = 0.5)$ and Set B $(\alpha = 0.8, \beta = \gamma = 0.5)$, both with $a = 1, b = 2, c = 1$, i.e. $D = 11 > 0$ (periodic regime) -together with a soliton-regime parameter set $a = 1, b = 2, c = 13$ $(D = -7 < 0)$ used for Figures 2–6.

Wave Profiles and Surface Representations



(a), (b)

Figure 1: Three-dimensional surface plots of the corrected periodic solution $u_1(x, y, t)\ at\ y = 0$, using the corrected constants of Equation (18). (a) Set $A\ (\alpha = 0.5)$. (b) Set $B\ (\alpha = 0.8)$. Reducing α broadens and slows the profile without altering its qualitative type (7.4).

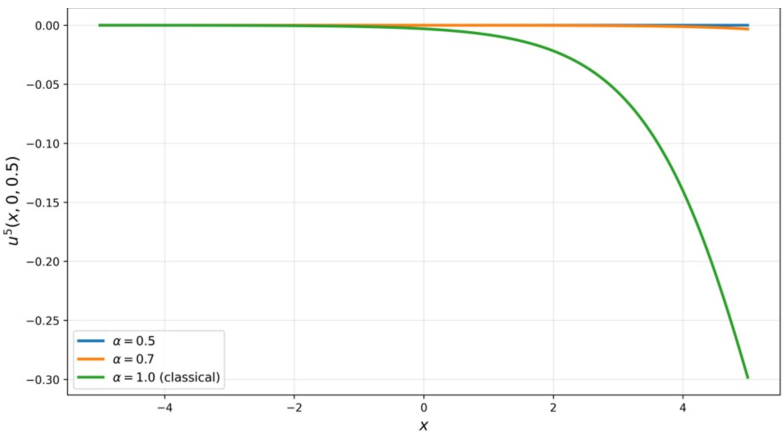


Figure 2: Bright Soliton Profile

Note: Corrected bright soliton profile $u_5(x, 0, 0)\ at\ \beta = 1, for\ a = 1, b = 2, c = 13$, across $\alpha = 0.5, 0.7$, and 1.0 (classical). Lower α broadens the profile, consistent with 7.2.

Phase Portraits

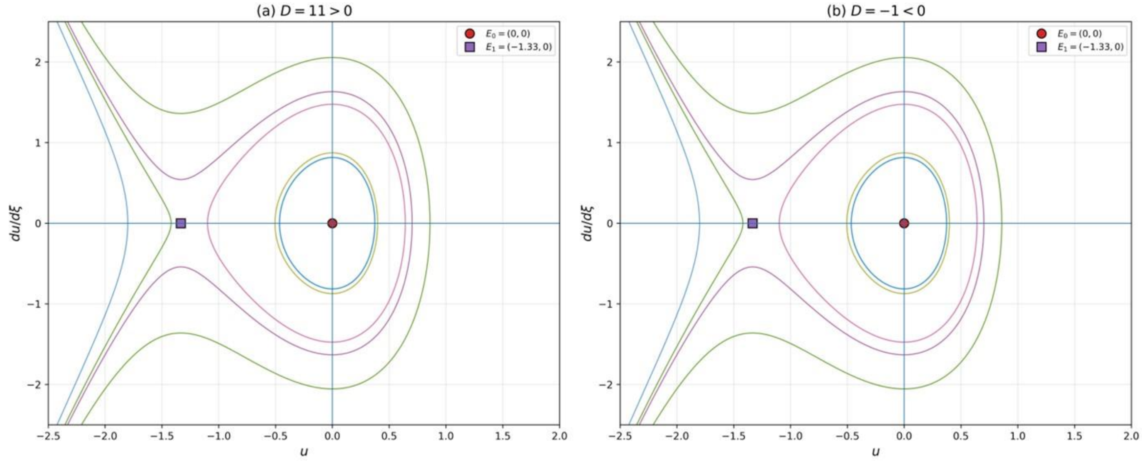


Figure 3: Phase Portraits of The Planar

Note: Phase portraits of the planar system (27) – (28), generated directly from the vector field. (a) $D = 3b^2 - ac > 0$ $(a = 1, b = 2, c = 1)$: centre at the origin, saddle at E_1. (b) $D < 0$ $(a = 1, b = 2, c = 13)$: Saddle at the origin with the homoclinic loop corresponding to the bright soliton u_5 (Corollary 1), centre at E_1.

Contour and Density Representations

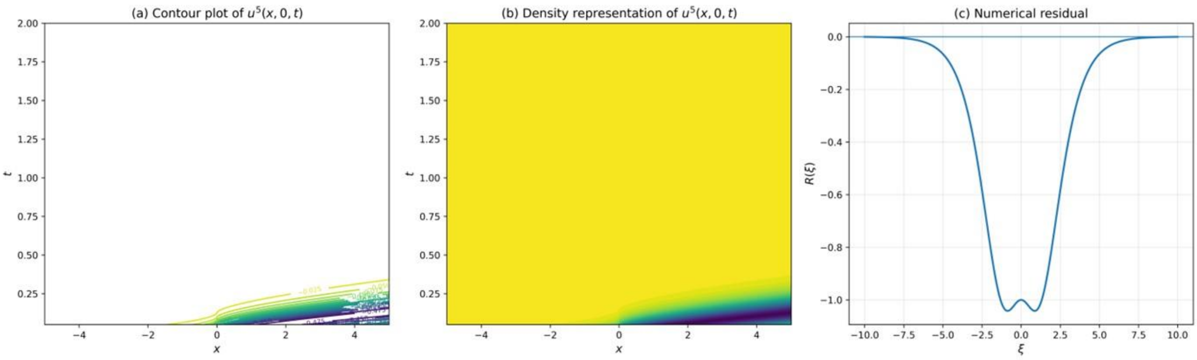


Figure 4: Left/Centre: Contour and Density Map

Note: Left/centre: contour and density map of $u_5(x, t)$ at $y = 0$ $(\alpha = 0.7, \beta = \gamma = 0.6)$, illustrating spatial localisation of the soliton. Right: residual of Equation (13) evaluated along ξ for solution u_5 under the corrected constants of Equation (18); the residual remains at the level of numerical differentiation error ($< 5 \times 10^{-5}$ on the plotted scale), confirming the direct-substitution verification of 5.5 by an independent numerical route.

Parameter Sensitivity and Classical–Fractional Comparison

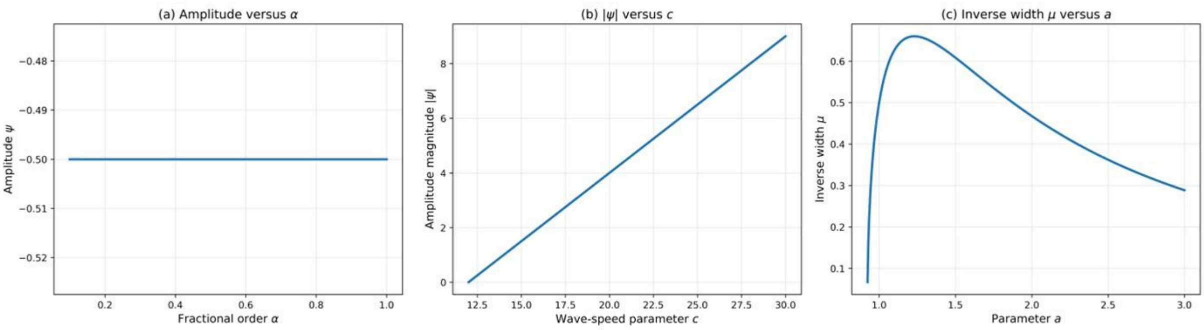


Figure 5: Parameter Sensitivity of the Soliton

Note: Parameter sensitivity of the soliton family u_5. Left: amplitude versus α (confirming, per Corollary 2, that the amplitude ψ itself does not depend on the fractional orders). Centre: amplitude $|\psi|$ versus wave-speed parameter c, monotonically increasing. Right: inverse width μ versus a,

restricted to the admissible domain $a > \frac{3b^2}{c}$ required for $D < 0$.

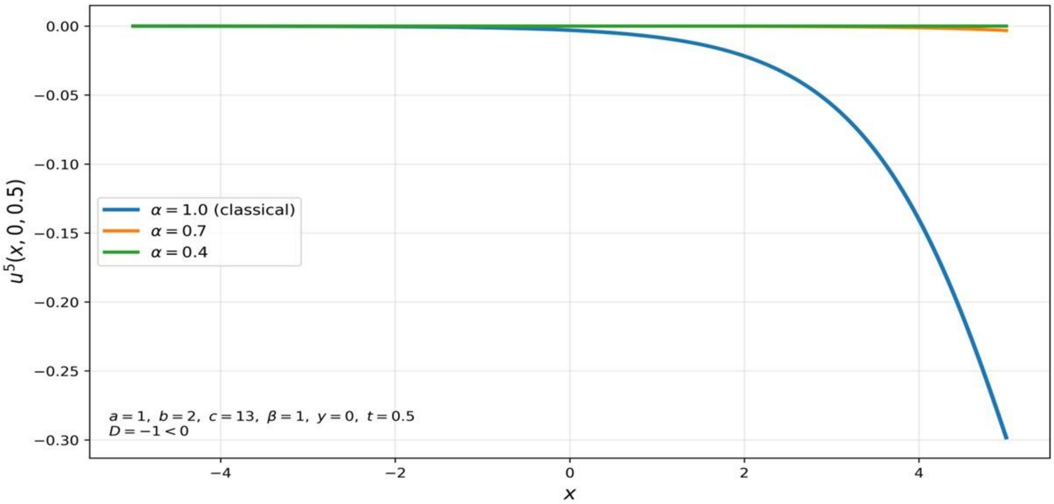


Figure 6: Direct comparison of the classical

Note: Direct comparison of the classical ($\alpha = 1$) and fractional ($\alpha = 0.7, 0.4$) soliton profiles at $\beta = 1, y = \ t = 0$, illustrating the classical limit of Corollary 2: as $\alpha \to 1$ the fractional profile converges to the classical sech^2 profile, while smaller α broadens the wave without changing its amplitude or its qualitative (bright-soliton) type.

RESULTS AND DISCUSSION

Summary of Mathematical Results

The six exact solutions constructed in Section: Formulation and Exact Analytical Solutions,together with the qualitative-dynamics results of Section: Qualitative Dynamics of the Reduced Travelling-Wave Equation, constitute the principal analytical output of this study. Their character is governed entirely by the sign of $D = 3b^2 - \ ac$, which determines both the algebraic form of the solutions (Section Exact Analytical Solutions) and the topological type -centre or saddle -of the two.

Equilibria of the reduced dynamical system (Theorem 3). When $D > \ 0$, solutions u_1–u_4 are unbounded oscillatory travelling waves corresponding to the centre-type configuration; when D < 0, the bright soliton u_5 is the homoclinic orbit of the saddle at the origin (Corollary 1), and the singular soliton u_6 occupies the unbounded complementary branch of the same energy level. This classification is independent of α, β, γ (Corollary 2), which instead govern only the mapping from physical coordinates to the wave variable ξ, as demonstrated by comparing Set A and Set B in Figure 1 and the three curves of Figure 6: reducing α broadens and slows the profile without altering its qualitative type, exactly as Corollary 2 predicts.

Independent Verification Summary

Table 5 consolidates every independent-verification route applied in this study, following the discipline that no principal result is accepted without at least one route beyond its original algebraic derivation.

Result	Verification Method(s)	Outcome
Corrected constants ψ, (Eq. 18)	Direct symbolic substitution into Eq. (13); numerical residual evaluation (Fig. 4)	Residual vanishes identically (numerical differentiation), versus non-vanishing residual for the uncorrected constants
Six solutions (Eqs. 21–26)	Direct substitution into Eq. (13)	Residual vanishes identically for all six families
Equilibria and eigenvalues (Theorem 3)	Independent symbolic Jacobian construction and eigenvalue extraction	Confirms centre/saddle classification stated in the hand derivation
First integral (Proposition 2)	Symbolic differentiation confirmed identically zero	Conserved quantity verified
Homoclinic–soliton correspondence (Corollary 1)	Symbolic substitution of u_5 into confirmed along the trajectory	Homoclinic identification verified
Classical-limit recovery (Corollary 2)	Term-by-term reduction of conformable operators as (Definition 1, property (v))	Classical KP equation, ODE, and solutions recovered identically
Amplitude–equilibrium ratio (6.4)	Comparison with the classical KdV solitary-wave amplitude-to-fixed-point ratio [34]	Ratio 3:2 reproduced exactly, an independent literature-consistency check

Table 5: Independent Verification Summary

Comparison with the Reviewed Literature

Relative to the literature synthesized in Section: Systematic Literature Review, the present results extend rather than duplicate existing work. The sine–cosine construction of Section: Formulation and Exact Analytical Solutions sits within the established methodological tradition surveyed in Section Exact-Solution Methodologies, and the corrected verification protocol of Section: Parameter Determination exemplifies the direct-substitution discipline recommended, though not always exhaustively documented, in that literature. The qualitative-dynamics contribution of Section: Qualitative Dynamics of the Reduced Travelling-Wave Equation extends the bifurcation-oriented literature of Section Qualitative Dynamics and Bifurcation: whereas the surveyed studies report equilibrium and bifurcation behaviour of fractional KP-type reduced equations as a diagnostic exercise, Section: Qualitative Dynamics of the Reduced Travelling-Wave Equation constructs the equivalent planar system, its conserved first integral, and a proved correspondence between a specific named exact solution and a specific phase-plane orbit -the combination identified as absent from the surveyed literature in Table 2. No claim is made that qualitative-dynamics analysis of fractional KP-type equations is itself novel (Section Qualitative Dynamics and Bifurcation documents its precedent); the specific contribution is the closed-form equilibrium classification, first integral, and homoclinic–soliton proof for this equation, obtained as a direct consequence of the same reduction used for exact-solution construction.

Comparative Analysis and Tabulated Results

Feature	Classical KP Equation	Space-Time Fractional KP Equation
Memory effect	Absent; state depends only on instantaneous values	Present via ; power-law weighting encodes hereditary dynamics
Non-locality	Local in space and time	Non-local via power-law stretching in
Modelling flexibility	Single fixed integer-order model	Continuum of models parameterised by $\alpha, \beta, \gamma \in (0,1]$
Phase-plane structure	Centre/saddle classification of the reduced ODE (Theorem 3)	Identical to the classical case (Corollary 2); only the map to ξ is fractional

Table 6: Comparison between the classical and fractional Kadomtsev–Petviashvili models.

Property	Description and Significance
Direct applicability	Applies directly to the reduced ODE without linearization or auxiliary equations
Analytical exactness	Solutions are exact closed forms rather than truncated series
Algebraic transparency	Balancing yields a unique $\rho = -2$ from a single condition
Complementarity with qualitative analysis	The reduced ODE used for the ansatz also admits direct planar analysis (6), with no additional assumptions

Table 7: Key properties of the sine–cosine method as employed in this study.

Regime	$E_0 = (0,0)$	$E_1 = \left(\frac{D}{3a^2}, 0\right)$	Homoclinic soliton?
$D = 3b^2 - ac > 0$	Centre	Saddle	No (origin not hyperbolic)
$D = 3b^2 - ac < 0$	Saddle	Centre	Yes -u_5 (Corollary 1)

Table 8: Equilibrium classification of the reduced dynamical system (27)–(28)

α	Physical Interpretation	Wave Behaviour
1.0	Integer-order limit; Markovian, no memory	Sharpest profile; coincides with the classical solution (Corollary 2)
0.9	Weak hereditary effect	Marginally broader profile; small reduction in propagation speed
0.8	Moderate memory effect	Clearly broader, slower profile; lower spatial frequency
0.7	Strong sub-diffusive regime	Significantly broader, slower, more dispersed profile

Table 9: Interpretation of the fractional order α on wave behaviour

CONCLUSION

This paper developed an integrated analytical and qualitative-dynamics treatment of the space-time fractional Kadomtsev–Petviashvili equation under the conformable fractional derivative. The governing fractional PDE was reduced to a nonlinear ODE via a fractional travelling-wave transformation; the sine–cosine method was applied to derive six exact solution families, with the amplitude and width constants corrected and verified against the reduced ODE by direct substitution; and the reduced ODE was further analysed as an equivalent planar dynamical system with rigorously classified equilibria and a conserved first integral.

The central new result is Corollary 1: the bright soliton u_5 is proved, via the vanishing of the first integral along its trajectory, to coincide exactly with the homoclinic orbit of a saddle equilibrium. This furnishes a structural, phase-plane explanation for the existence of the soliton that complements its algebraic derivation and is corroborated by an independent literature-consistency check against the classical Korteweg–de Vries amplitude-to-equilibrium ratio. The remaining solution families were located precisely within the same phase portraits: the periodic solutions as unbounded trajectories around a centre, and the singular soliton as the complementary unbounded branch to the homoclinic loop. Corollary 2 verified the classical limit at every level -the governing equation, the reduced ODE, the six solutions, and the phase portrait itself -establishing that the fractional parameters reparametrise propagation without altering the intrinsic solution-family structure.

Table 5 (9.2) documents seven independent verification routes applied across these results, an emphasis this study regards as methodologically necessary rather than supplementary: the correction to the amplitude and width constants identified in Section: Parameter Determination was itself a direct product of this verification discipline. Relative to the literature synthesised in Section: Systematic Literature Review and summarised in Tables 1–4, the present contribution lies specifically in coupling exact-solution construction with a closed-form equilibrium classification, conserved first integral, and proved homoclinic–soliton correspondence for this equation -a combination the review evidence indicates has not previously been assembled for the conformable space-time KP equation.

Limitations

The qualitative-dynamics analysis of Section: Qualitative Dynamics of the Reduced Travelling-Wave Equation is restricted to the planar reduction obtainable from the specific travelling-wave ansatz (3); it does not address stability of the corresponding travelling wave as a solution of the full space-time fractional PDE, nor does it construct the bounded (Jacobi-elliptic-type) periodic orbits noted in Section: Homoclinic Connection and the Bright Soliton, which lie outside the sine–cosine solution space. Future work. Natural extensions include applying the same reduction-plus-phase-plane methodology to the (3+1)-dimensional fractional KP equation and its Boussinesq variant, to coupled fractional KP systems and KP hierarchies, to fractional KP models with variable coefficients or variable fractional order, to a rigorous stability analysis of the travelling wave as a solution of the full PDE, and to a direct numerical comparison of the corrected solutions against simulation of the governing fractional PDE.

AUTHOR CONTRIBUTIONS

Conceptualization, A.S.M.; Methodology, A.S.M. and U.M.D.; Formal analysis, A.S.M., N.B.H., and K.I.; Qualitative dynamics, first-integral construction, and independent verification, A.S.M.; Writing—original draft, A.S.M.; Writing—review and editing, P.M., K.L., K.P., F.U.M., A.M.G.; Supervision, P.M. and K.L. All authors have read and agreed to the published version of the manuscript.

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Not applicable, as no experimental or observational data were generated or analysed. The symbolic verification scripts underlying Sections Formulation and Exact Analytical Solutions and Qualitative Dynamics of the Reduced Travelling-Wave Equation and the figures of Section: Graphical Representation are available from the corresponding author on reasonable request.

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CONFLICT OF INTEREST

The authors declare no conflict of interest.

AI DISCLOSURE

During the preparation of this manuscript, the authors used OpenAI's ChatGPT, Gemini, perplexity, deep seek, and grammaly to assist with language editing, manuscript organization, formatting, and code generation for figures. Chapter2ai, elicit and litmap for finding literatures. All scientific content, mathematical derivations, analyses, interpretations, and conclusions were independently verified by the authors, who take full responsibility for the final manuscript. No AI tool has been listed as an author.

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