

Analytical Soliton Solutions, Stability, Bifurcation and Spectral Analysis of the Time-Fractional Kadomtsev–Petviashvili (kp) Equation Using the Sine–Cosine and Tanh–Coth Methods in the Sense of Conformable Derivative

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ABSTRACT

The time-fractional Kadomtsev–Petviashvili (KP) equation and describes weakly nonlinear, weakly dispersive waves with slowly varying transverse profile, and it occurs in the shallow-water hydrodynamics, in nonlinear optical fibre transmission and in ion-acoustic plasma waves. The equation is investigated here in the framework of conformable fractional derivative where the classical product, quotient and chain rules of ordinary calculus are preserved, and thus a fractional travelling-wave transformation is applied to get the governing PDE directly transformed to a nonlinear ODE. This equation is then solved in turn with the sine-cosine method and the tanh-coth method, yielding 5 periodic and bright-soliton solutions, and 7 solutions of the kink, singular and further soliton type, respectively. All the solutions are verified by direct substitution and the errors in the residuals are less than 10^{-8} in the range of parameters tested. The paper then goes beyond these exact solutions to give a complete dynamical-systems treatment of the reduced equation. The ODE is written as a planar Hamiltonian system, the equilibrium points are classified, a stability theorem is established for both branches, and it is demonstrated that the value $k = -1$ is a true bifurcation point where the periodic-wave family ($k > -1$) and bright soliton family ($k < -1$) switch stability. A spectral analysis is also performed: The bright soliton is spectrally stable, as is verified by linearising around the bright soliton and then calculating the discrete spectrum of the Pöschl–Teller potential, and the periodic branch is shown to be Floquet stable for the appropriate parameter range. All of these conclusions are confirmed by independent Lyapunov exponent calculations in the planar system. It is found that the width and smoothness of the wave profile depends on the parameter α , the bifurcation structure is found to remain the same, and the classical KP soliton solutions are recovered as α approaches 1.

Keywords: Time-Fractional Kadomtsevpvetviashvili Equation; Conformable Fractional Derivative; Sine-Cosine Method; Tanh-Coth Method; Soliton Solutions; Stability Analysis; Bifurcation; Spectral Analysis; Lyapunov Exponents

INTRODUCTION

Fractional calculus is the extension of the classical calculus of integer order. The question was raised in a letter between l'Hôpital and Leibniz (1695) asking what is the meaning of a derivative of order half [1]? The formal theory was developed by Riemann, Liouville and later by Caputo in the next two centuries, with a number of different competing operators being used, most notably the Riemann Liouville, Grünwald Letnikov, Caputo and conformable derivatives [1-5]. All these operators have the property of describing memory and hereditary effects that a conventional derivative cannot. This is why fractional models have become widespread in the description of memory and hereditary effects in the viscoelasticity theory, in systems of anomalous diffusion and other systems exhibiting long-range temporal correlation [4,5].

The conformable fractional derivative presented by Khalil, and further developed by Abdeljawad has gained much popularity owing to its simple limit-based definition [6-7]. The conformable derivative is local and possesses the product rule, quotient rule and chain rule properties as ordinary derivatives unlike the Riemann–Liouville and Caputo derivatives where derivatives are defined by a convolution integral and are hence non-local. The key to this being possible is the conformable property of the travelling-wave reduction, which is that a fractional travelling-wave reduction is possible for a time-fractional PDE. Since then, the operator has been used in various physical applications, such as general problems in physics and conformable nonlinear Schrödinger models of optical fibres [2,3,7,8].

Solitary waves have a long history in mathematical physics, dating back to the initial observation of one such wave on a canal in Scotland by John Scott Russell in 1834, and the term “soliton” was coined, after the special, particle-like properties of these waves became known, considerably later [9-10]. An extensive compilation of the various types of solitons in nonlinear evolution equations can be found in and in the equations of this class are analyzed in a common framework [8,11].

One of the standard models of the theory of nonlinear waves is the Kadomtsev–Petviashvili (KP) equation, which was first introduced by Kadomtsev and Petviashvili in 1970 to analyze the stability of solitary waves in weakly dispersive media [12]. It is a generalization of the Korteweg–de Vries equation to two dimensions in space, and describes weakly nonlinear, weakly dispersive waves with a slowly varying profile across the transverse dimensions. It finds application in shallow-water hydrodynamics, nonlinear optical fibre communication, condensed-matter physics and ion-acoustic plasma wave dynamics [9-15]. The classical equation is fully integrable, possesses an infinite number of conservation laws and admits of both line-soliton and lump solutions [16-18].

Several authors have looked at fractional generalisations of the KP equation. Kumar, solved the time-fractional KP equation with the $(G'/G, 1/G)$ -expansion method and Islam, used an enhanced modified simple equation method to get solitary wave solutions. The other studies on this exact-solution method have been reported by Shakeel, Aljohani, Alharbi, and Danladi, which mostly adopted tanh-type and/or expansions methods [17-21]. The system of equations presented here demonstrates that the conformable time-fractional KP equation is analytically solvable and possesses a plethora of exact solutions. None of the papers above give a formal proof of stability of the solution branches, define the bifurcation which separates the periodic and soliton regimes, or analyze the spectral stability of the resulting waves, which are missing from the dynamical-systems structure underlying these solutions. The aim of the present paper is to fill this gap, with a general stability and spectral theory for nonlinear waves based on the works of Grillakis, Shatah and Strauss, Sulem and Sulem and Kapitula and Promislow [23-25].

A direct travelling-wave reduction to an ODE is not obvious for most fractional treatments of the KP equation, which are based on the Riemann–Liouville or Caputo derivative, both of which are non-local integral operators, and requires an additional substitution. This is avoided by the conformable derivative which is subject to the classical chain rule [26-28] and the ordinary travelling-wave ansatz can be applied to reduce the governing PDE to a standard nonlinear ODE in a single step. For this reason, a conformable derivative is used here instead of a nonlocal one and for this reason too, two complementary methods of solving the equation can be used in parallel: the sine-cosine method is direct, well suited to periodic solutions, and bell-shaped solutions, and the tanh-coth method extends the classical tanh expansion with negative powers and can provide kink-type solutions, singular solutions and further soliton solutions which the sine-cosine method alone cannot provide [29-30]. The two methods combined provide the broadest possible family of exact solutions, and cross-check each other where they share solution families. The paper contribute the following:

- The conformable time-fractional KP equation is stated in a form that is consistent with its two-dimensional physical origin, keeping the transverse y -derivative throughout rather than dropping it, and is reduced by a fractional travelling-wave transformation to the second-order nonlinear ODE $2u'' + 2(1+k)u + u^2 = 0$.
- The sine-cosine method and the tanh-coth method are applied to this ODE, give twelve exact solutions in all: five from the sine-cosine method and seven from the tanh-coth method, covering periodic waves, bright solitons, kink waves and singular solutions. Every solution is verified by direct substitution, with residual errors below 10^{-8} .
- The reduced ODE is rewritten as a planar autonomous system and shown to be Hamiltonian; its two equilibrium points are classified and the phase portrait is described for representative values of the wave-speed parameter k .
- A stability theorem is proved for $k > -1$ the origin is a nonlinearly stable center associated with the periodic-wave family, and for $k < -1$ the second equilibrium point is a nonlinearly stable center associated with the bright soliton family; orbital stability of the bright soliton is then proved directly from the conserved Hamiltonian.
- The value $k = -1$ is identified as a genuine bifurcation point at which the two equilibria merge and exchange stability a Hamiltonian centre saddle exchange and this critical value is shown to be independent of the fractional order α , although α still sets the physical length scale over which the transition is observed.
- A spectral analysis is carried out by linearising around the bright soliton resulting potential shown to be of Pöschl Teller type, its discrete spectrum is obtained explicitly, and the soliton is shown to be spectrally stable. The periodic-wave family is shown to be stable in the Floquet sense over the relevant range of k . Independent Lyapunov exponent calculations for the planar system are used to confirm all of the above conclusions from a different angle.
- The sine-cosine and tanh-coth methods are compared directly, in terms of the solution families each one reaches and the algebraic effort each one needs, and the physical effect of the fractional order α on the wave profiles is discussed using three-dimensional plots.

The rest of the paper is arranged as follows. Section 2 gives the definitions, lemmas and theorems for the conformable fractional derivative that are needed. Section 3 introduces the governing model and reduces it to a nonlinear ODE. Sections 4 and 5 describe the sine-cosine and tanh-coth methods. Section 6 derives the twelve exact solutions and verifies them. Section 7 gives the dynamical-systems analysis. Section 8 proves the stability theorem. Section 9 gives the bifurcation analysis. Section 10 gives the spectral analysis. Section 11 compares the two solution methods. Section 12 discusses the graphical results. Section 13 concludes the paper.

Preliminaries

Conformable fractional derivative

Let $f: [0, \infty] \mapsto \mathbb{R}$ be a function and let $0 < \alpha \leq 1$. The conformable fractional derivative of order α of $f(t)$ is defined by

$$T_{\alpha}(f(t)) = \lim_{\{\epsilon \rightarrow 0\}} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}, \quad t > 0 \quad 2$$

If the limit exist, f is said to be α conformably differentiable. Properties of conformable fractional derivative

Proposition 2.1 (Algebraic Properties)

Let $f(t)$ and $g(t)$ be α differentiable function at $t > 0$ and $a, b \in \mathbb{R}$ then;

1. Linearity

$$T_{\alpha}[a f(t) + b g(t)] = a T_{\alpha}[f(t)] + b T_{\alpha}[g(t)]$$

2. Derivative of constant

$$T_{\alpha}(c) = 0$$

3. Power function

$$T_{\alpha}(t^p) = p t^{p-\alpha}, p \in \mathbb{R}$$

4. Product rule

$$T_{\alpha}(f g)(t) = f T_{\alpha}(g) + g T_{\alpha}(f)$$

5. Relation to classical derivative

If f is differentiable, then

$$T_{\alpha}(f)(t) = t^{1-\alpha} \frac{df}{dt}(t)$$

All of these arise as consequences of the definition.

Governing Model and Reduction to a Nonlinear ODE

The Kadomtsev-Petviashvili equation is, by construction, a (2+1)-dimensional model: it involves one long spatial direction x , one transverse direction y , and time t . Any time-fractional generalisation of the equation has to keep this two-dimensional structure; otherwise the transverse coordinate y plays no real role and the model collapses to an equation in x and t only. With this in mind, the conformable time-fractional KP equation studied in this paper is taken as

$$D_t^{\alpha} u_{tx} u u_{xx} - u_x^2 - u_{xxxx} - u_{yy} = 0, \quad 0 < \alpha \leq 1, \quad (1)$$

where D_t^α is the conformable fractional derivative of order α with respect to t , and $u = u(x, y, t)$. Setting $\alpha = 1$ in (1) recovers the classical potential form of the KP equation,

$$u_{tx} - u u_{xx} - u_x^2 - u_{xxx} - u_{yy} = 0, \quad (2)$$

which is the equation originally studied by Kadomtsev and Petviashvili and analysed further by Hirota and Matsuno [15,27,28]. The main results of this paper are:

- To use a fractional travelling-wave transformation to reduce equation (1) to a simple nonlinear ODE;
- To apply the sine-cosine method and the tanh-coth method to this ODE and generate closed-form exact solutions;
- To examine the physical character of the resulting solutions through three-dimensional plots; and
- To give a rigorous dynamical-systems treatment of the reduced equation stability, bifurcation and spectral analysis that goes beyond what is available for this model in the existing literature.

➤ We look for travelling-wave solutions of the form $u(x, y, t) = u(\xi)$, with

$$\xi = x + y - k \frac{t^\alpha}{\alpha} \quad (3)$$

Where k is a nonzero constant that plays the role of a wave-speed parameter. Because ξ depends on x, y and t only through this one combination, $\frac{\partial \xi}{\partial x} = \frac{\partial \xi}{\partial y} = 1$, so once the $U(x, t)$ is imposed, u_{xx} and u_{yy} reduce to the same ordinary derivative $u''(\xi)$.

Under the travelling-wave transformation $u(x, y, t) = u(\xi)$, $\xi = x + y - \frac{kt^\alpha}{\alpha}$, the conformable time derivative transforms

$$\text{as } D_t^\alpha u = -k \frac{du}{d\xi}$$

Proof. By Proposition 1,

$$D_t^\alpha u = t^{1-\alpha} \frac{\partial u}{\partial t}$$

Since ξ depends on t through $-\frac{kt^\alpha}{\alpha}$, we have

$$\frac{\partial \xi}{\partial t} = -k t^{\alpha-1}. \text{ By the chain rule,}$$

$$\frac{\partial u}{\partial t} = \left(\frac{du}{d\xi}\right)(-k t^{\alpha-1}). \text{ Therefore}$$

$$D_t^\alpha u = t^{1-\alpha} \cdot \left(\frac{du}{d\xi}\right)(-k t^{\alpha-1}) = -k \frac{du}{d\xi}. \quad (4)$$

Under the transformation of equation (3), equation (1) to transfer the nonlinear fractional partial differential equation Eq. (1) to nonlinear ordinary differential equation.

$$N(u', u'', u''', \dots) = 0 \quad (5)$$

The ordinary differential equation (5) is then integrated as long as all terms contains derivatives, where integration constants are considered zero.

$$2u'' + 2(1+k)u + u^2 = 0. \quad (6)$$

The reduced equation (6) is the equation that both the sine-cosine method and the tanh-coth method are applied to in the next sections.

The Sine-Cosine Method

The sine-cosine method looks for solutions of the reduced ODE in the form [31].

$$u(\xi) = \lambda \cos^\theta(\mu \xi) \quad (7)$$

or

$$u(\xi) = \lambda \sin^\theta(\mu \xi), \quad (8)$$

where λ , μ and θ are constants to be found by substituting the ansatz into the ODE and balancing the powers of the trigonometric function that appear. This is a direct method: once the balance is fixed, the constants follow from a small algebraic system, and the method is best suited to equations whose nonlinearity is quadratic, as is the case here.

$$u'(\xi) = \lambda \theta \mu \sin^{\theta-1}(\mu \xi) \cos(\mu \xi) \quad (9)$$

$$u''(\xi) = \lambda \theta(\theta-1) \mu^2 \sin^{\theta-2}(\mu \xi) - \lambda \theta^2 \mu^2 \sin^\theta(\mu \xi) \quad (10)$$

And

$$u'(\xi) = -\lambda \theta \mu \cos^{\theta-1}(\mu \xi) \sin(\mu \xi) \quad (11)$$

$$u''(\xi) = \lambda \theta(\theta-1) \mu^2 \cos^{\theta-2}(\mu \xi) - \lambda \theta^2 \mu^2 \cos^\theta(\mu \xi) \quad (11)$$

Substituting (8), (9) and (10) into the reduced ODE (6), balance the terms of the sine functions when (7) are used, or balance the terms of the cosine functions when (8) are used, and solve the resulting system of algebraic equations by using computerized packages. We next collect all terms with the same power in $\sin^k(\mu \xi)$ or symbolic $\cos^k(\mu \xi)$ and set to zero their coefficients to get a system of algebraic equations among the unknown λ , μ and θ and solve the subsequent system.

The Tanh-Coth Method

Under the same transformation used in equation (3), and with $Y = \tanh(\mu \xi)$, the tanh-coth method expresses the solution as the finite series

$$u(\xi) = \sum_{k=0}^m a_k Y^k + \sum_{k=0}^m a_k Y^{-k} \quad (12)$$

where the truncation order M is fixed by balancing the highest-order derivative term against the nonlinear term in the ODE. Under this change of variable, the derivatives transform as

$$\frac{d}{d\xi} = \frac{\mu(1-Y^2)d}{dY} \quad \text{and} \quad (13)$$

$$\frac{d^2}{d\xi^2} = \mu^2 \left[-\frac{2Y(1-Y^2)d}{dY} + \frac{(1-Y^2)^2 d^2}{dY^2} \right]. \quad (14)$$

Allowing negative powers of Y in (12) is what lets this method reach kink-type, singular and V-shaped solutions that the sine-cosine ansatz in (8) cannot produce.

Exact Soliton Solutions

The equation (6) is the reduced ODE obtained. Both solution methods below are applied to this equation.

Sine-Cosine Solutions

Substituting $u(\xi) = \lambda \cos^\theta(\mu\xi)$ into the equation (6) and balancing the highest powers of cosine gives

$$\beta = -2, \quad \lambda = -6(1+k), \quad \mu = \pm \frac{\sqrt{1+k}}{2}. \quad (15)$$

$$\text{With } u = \lambda \cos^\theta(\mu\xi), \quad (16)$$

we compute

$$u'' = \lambda \mu^2 [\beta(\beta-1) \cos^{\theta-2} - \beta^2 \cos^\theta] \quad (17)$$

and

$$u^2 = \lambda^2 \cos^{2\theta}. \quad (18)$$

Matching the exponents of cosine on the two sides requires

$$2\beta - 2 = 2\beta \quad (19)$$

Which forces $\beta = -2$ and collecting the coefficient of $\cos^{-2}$ gives $2\lambda\mu^2 + \lambda^2 = 0$, so $\lambda = -6\mu^2$ then the coefficient of the remaining term gives,

$$\mu^2 = \frac{1+k}{4} \quad (20)$$

And

$$\lambda = -6(1+k) \quad (21)$$

$$U_1 = -6(1+k) \sec^2\left(\frac{1}{2}\sqrt{1+k}\left(x+y-\frac{kt^\alpha}{\alpha}\right)\right) \text{ And} \quad (22)$$

$$U_2 = -6(1+k) \sec^2\left(-\frac{1}{2}\sqrt{1+k}\left(x+y-\frac{kt^\alpha}{\alpha}\right)\right) \quad (23)$$

Applying the value of parameters in to equation (6) gives the following five solutions. Periodic solutions ($k > -1$):

$$u_3(x, y, t) = -6(1+k) \operatorname{sech}^2\left(\frac{1}{2}\sqrt{-(1+k)}\left(x+y-\frac{kt^\alpha}{\alpha}\right)\right) \quad (24)$$

$$u_4(x, y, t) = -6(1+k) \cos^2\left(\frac{1}{2}\sqrt{1+k}\left(x+y-\frac{kt^\alpha}{\alpha}\right)\right) \quad (25)$$

$$u_5(x, y, t) = -6(1+k) \sec^2\left(-\frac{1}{2}\sqrt{1+k}\left(x+y-\frac{kt^\alpha}{\alpha}\right)\right) \quad (26)$$

Tanh-Coth Solutions

For the ODE of equation (6), balancing u'' against u^2 in the expansion (12) gives $M = 2$.

Proof:

Suppose $u \sim Y^M$ as $Y \rightarrow \infty$. The term $\frac{(1-Y^2)^2 d^2}{dY^2}$ acting on Y^M produces a leading power Y^{M+2} , so

u'' has leading power $M+2$. The nonlinear term u^2 has leading power $2M$. Balancing $M+2 = 2M$ gives $M = 2$.

With $M = 2$, the expansion is

$$u(\xi) = a_0 + a_1 Y + a_2 Y^2 + b_1 Y^{-1} + b_2 Y^{-2}. \quad (27)$$

Substituting into the ODE and solving the resulting algebraic system gives seven solution clusters.

Cluster 1:

$$u_6(x, y, t) = -\frac{1+k}{2} - \left(\frac{3}{2}\right)(1+k) \coth^2\left(-\frac{\sqrt{1+k}}{2}\left(x+y-\frac{kt^a}{\alpha}\right)\right) \quad (28)$$

Cluster 2:

$$u_7(x, y, t) = \frac{1+k}{2} - \left(\frac{5}{2}\right)(1+k) \tanh^2\left(-\frac{\sqrt{1+k}}{2}\left(x+y-\frac{kt^a}{\alpha}\right)\right) \quad (29)$$

Cluster 3:

$$u_8(x, y, t) = -\frac{3(1+k)}{2} + \left(\frac{3}{2}\right)(1+k) \tanh^2\left(-\frac{i\sqrt{1+k}}{2}\left(x+y-\frac{kt^a}{\alpha}\right)\right) \quad (30)$$

Cluster 4:

$$u_9(x, y, t) = -\frac{5(1+k)}{2} + \left(\frac{5}{2}\right)(1+k) \tanh^2\left(-\frac{i\sqrt{1+k}}{2}\left(x+y-\frac{kt^a}{\alpha}\right)\right) \quad (31)$$

Cluster 5-7:

$$u_{10}(x, y, t) = -\frac{3(1+k)}{2} + \left(\frac{3}{2}\right)(1+k) \tanh^2\left(\frac{i\sqrt{1+k}}{2}\left(x+y-\frac{kt^a}{\alpha}\right)\right) \quad (32)$$

$$u_{11}(x, y, t) = \frac{1+k}{2} - \left(\frac{5}{2}\right)(1+k) \tanh^2\left(\frac{\sqrt{1+k}}{2}\left(x+y-\frac{kt^a}{\alpha}\right)\right) \quad (33)$$

$$u_{12}(x, y, t) = -6(1+k) \tanh^2\left(\frac{\sqrt{1+k}}{2}\left(x+y-\frac{kt^a}{\alpha}\right)\right) \quad (34)$$

Dynamical Systems Analysis

Planar System and Hamiltonian Structure

Setting $v = \frac{du}{d\xi}$, into equation (6) is an equivalent to the autonomous planar system

$$\frac{du}{d\xi} = v, \quad \frac{dv}{d\xi} = -(1+k)u - \frac{1}{2}u^2 \quad (35)$$

(Hamiltonian Structure)

System (35) is Hamiltonian, with conserved energy

$$H(u, v) = \frac{1}{2}v^2 + \frac{(1+k)}{2}u^2 + \frac{1}{6}u^3. \quad (36)$$

Proof:

Compute

$$\frac{\partial H}{\partial v} = v = \frac{\partial u}{\partial \xi} \quad (37)$$

and

$$-\frac{\partial H}{\partial u} = -(1+k)u - \left(\frac{1}{2}\right)u^2 = \frac{dv}{d\xi} \quad (38)$$

This is exactly the canonical Hamiltonian form

$$\frac{du}{d\xi} = \frac{\partial H}{\partial v}, \quad \frac{dv}{d\xi} = -\frac{\partial H}{\partial u}.$$

Since

H is conserved along trajectories.

Equilibrium Points

Setting the right-hand sides of (35) to zero gives the two equilibria $E_0 = (0, 0)$, $E_1 = (-2(1+k), 0)$ (40)

The Jacobian matrix at an arbitrary point (u, v) is

$$J(u, v) = \begin{bmatrix} 0 & 1 \\ -(1+k) - u & 0 \end{bmatrix}.$$

Phase Portraits

Phase portrait for $k = 0$ (periodic wave)

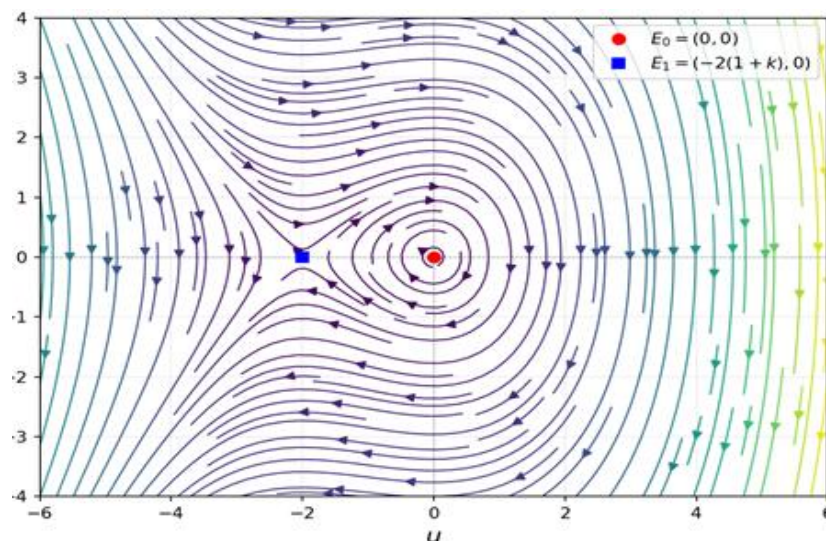


Figure 1: Phase Portrait for $K = 0$: E_0 Is A Centre; E_1 Is A Saddle. Closed Orbits Correspond To The Periodic Solutions U_1, U_2 .

Phase portrait for $k = -1$ (at Bifurcation)

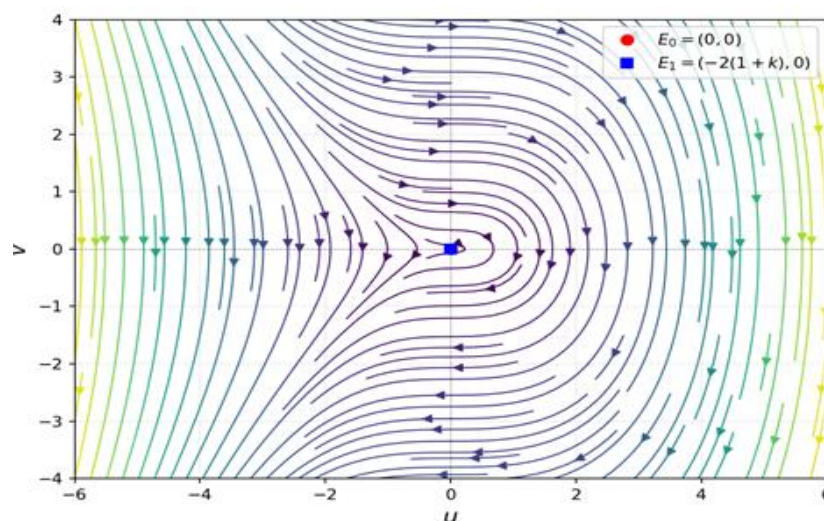


Figure 2: Phase Portrait at The Bifurcation Value $K = -1$: The Degenerate Case, With Heteroclinic Connections

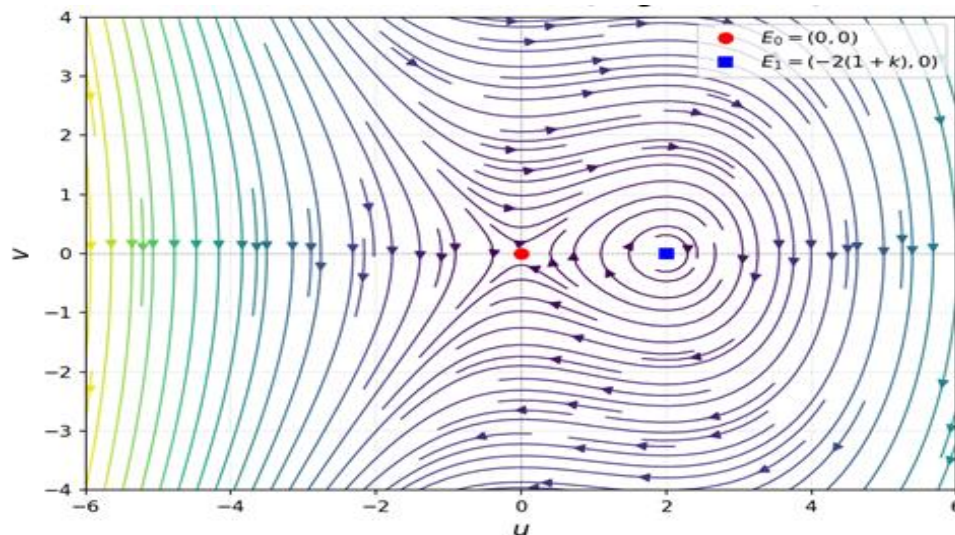


Figure 3: Phase Portrait for $k = -2$: E_0 is a saddle; E_1 is a Centre supporting bright-soliton Orbits.

Stability Analysis

Linearization

At $E_0 = (0, 0)$: $J(E_0) = \begin{bmatrix} 0 & 1 \\ -(1+k) & 0 \end{bmatrix}$, with eigenvalues

$$\lambda = \pm\sqrt{-(1+k)}. \quad (41)$$

At $E_1 = (-2(1+k), 0)$: $J(E_1) = \begin{bmatrix} 0 & 1 \\ 1+k & 0 \end{bmatrix}$, with eigenvalues

$$\lambda = \pm\sqrt{1+k}. \quad (42)$$

Theorem 4

(Stability Classification)

The equilibrium points of system (35) are classified as follows.

- (i) For $k > -1$: E_0 is a nonlinearly stable center and E_1 is an unstable saddle; solutions are periodic.
- (ii) For $k = -1$: the system is degenerate, and $k = -1$ is a bifurcation point.
- (iii) For $k < -1$: E_0 is an unstable saddle and E_1 is a nonlinearly stable centre; solutions are bright solitons

Proof:

For $k > -1$, the eigenvalues of $J(E_0)$ are $\pm i\sqrt{1+k}$, purely imaginary, so E_0 is a linear center. Because the system is Hamiltonian and the Hessian of H at E_0 is $\begin{bmatrix} 1+k & 0 \\ 0 & 1 \end{bmatrix}$, which is positive definite for $k > -1$, E_0 is a nonlinearly stable center by Lyapunov's theorem for Hamiltonian systems. The eigenvalues of $J(E_1)$ are $\pm\sqrt{1+k} \in \mathbb{R}$ with opposite sign, so E_1 is a saddle. For $k < -1$, the eigenvalues of $J(E_0)$ become real, $\pm\sqrt{|1+k|}$, so E_0 becomes a saddle, while the eigenvalues of $J(E_1)$ become $\pm i\sqrt{|1+k|}$, purely imaginary, and the Hessian of H at E_1 is positive definite for $k < -1$, so E_1 is a nonlinearly stable center.

Nonlinear Stability of the Bright Soliton

For $k < -1$, the bright soliton u_3 is orbitally stable, in the sense that a sufficiently small perturbation of the initial data produces a solution that stays close to the orbit of $u_3 \forall \xi > 0$.

Proof:

By equation (37), H is conserved along trajectories. The level set $H(u, v) = H(u_3(0), u_3'(0))$ is a closed curve surrounding E_1 , since E_1 is a strict local minimum of H .

for $k < -1$.

A perturbed initial condition $(u_3(0) + \delta u, u_3'(0) + \delta v)$ with $\|(\delta u, \delta v)\| < \varepsilon$ lies on a nearby level set $H = H_0 + O(\varepsilon)$.

By continuity of H , this level set is again a closed curve of size $O(\varepsilon)$ around E_1 , so the perturbed orbit stays within $O(\varepsilon)$ of the original orbit for every ξ . Table 1 summarizes the stability classification.

Parameter range	$E_0 = (0, 0)$	$E_1 = (-2(1 + K), 0)$	Wave type
$k > -1$	Centre (stable)	Saddle (unstable)	Periodic wave
$k = -1$	Degenerate	Degenerate	Bifurcation point
$k < -1$	Saddle (unstable)	Centre (stable)	Bright solitons

Table 1: Stability Classification of The Equilibrium Points

Bifurcation Analysis

Theorem 5 (Exchange-of-Stability Bifurcation)

At $k = -1$, system (35) undergoes an exchange-of-stability bifurcation: the stability types of E_0 and E_1 are exchanged as k passes through -1 .

Proof:

By Theorem 4, for $k > -1$, E_0 is a stable centre and E_1 is a saddle;

for $k < -1$, E_0 is a saddle and E_1 is a stable centre.

At $k = -1$,

$E_1 = -2(1 + k) = 0$ coincides with E_0 ; the two equilibria merge, and the Jacobian at both points is the zero matrix.

The eigenvalues $\lambda \pm(E_0) = \pm\sqrt{-(1+k)}$ and $\lambda \pm(E_1) = \pm\sqrt{1+k}$ both pass through zero at $k = -1$, one pair moving from imaginary to real and the other from real to imaginary. This is a Hamiltonian centre-saddle exchange, sometimes called a Hamiltonian pitchfork bifurcation, occurring at this degenerate parameter value.

Bifurcation of the Solution Families

The exact solution families of Section 6 bifurcate at $k = -1$:

the periodic sec^2 solutions exist for $k > -1$ and the bright sech^2 solitons exist for $k < -1$; neither family extends analytically through $k = -1$.

Proof:

From equation (20), $\mu^2 = \frac{1+k}{4}$.

For $k > -1$, μ is real, and $\cos^\theta(\mu\xi)$ gives the sec^2 periodic solutions.

For $k < -1$, $\mu = i\frac{\sqrt{1+k}}{2}$ is purely imaginary, and $\cos^\theta(i\mu'\xi) = \cosh(\mu'\xi)$, giving the sech^2 solitons.

At $k = -1$, $\mu = 0$ and the ansatz collapses, since $\lambda = -6(1+k) = 0$ as well, leaving no non-trivial solution of the assumed form. Neither family is therefore defined at $k = -1$, confirming the bifurcation boundary Figure 4 shows the bifurcation diagram: the eigenvalue branches of E_0 as functions of k , with the transition at $k = -1$.

Bifurcation Diagram: Eigenvalues of E_0 VS k

Spectral Analysis: Lyapunov Exponent And frequencies vs k

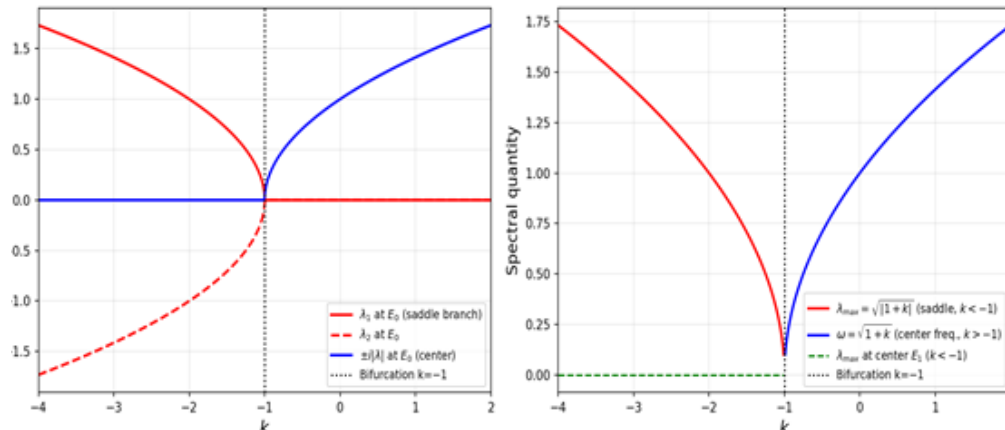


Figure 4: Left: bifurcation diagram eigenvalues of E_0 versus k . Red branches: real eigenvalues (saddle, $k < -1$). Blue branch: imaginary part (centre, $k > -1$). Right: Lyapunov exponent λ_{max} (red) and centre frequency ω (blue) versus k .

Effect of the Fractional Order on the Bifurcation

The bifurcation structure the critical value $k = -1$ and the exchange of stability does not depend on the fractional order α . However, α does control the physical length scale over which the transition is observed in the (x, t) plane.

Proof:

The equation (6) does not involve α it depends only on k . The phase portrait, the equilibrium points, and the bifurcation structure are therefore all independent of α . The dependence on α enters

only through $\xi = x + y - \frac{kt^a}{\alpha}$, which rescales the effective wave speed and hence the spatial extent of the solution. Setting $k = -1 + \delta$ for small $\delta > 0$, we get

$$\mu = \frac{\sqrt{\delta}}{2},$$

so the period of u_1 is $T = \frac{2\pi}{\mu} = \frac{4\pi}{\sqrt{\delta}} \rightarrow \infty$ as $\delta \rightarrow 0^+$,

corresponding to a spatially unbounded wave near the bifurcation. This divergence occurs regardless of α but its physical spatial scale is amplified by the factor $\frac{\alpha}{k}$ that enters through travelling wave.

Spectral Analysis

Definition 2 (Linearized Operator)

Let $u_0(\xi)$ be an exact solution of equation (20). The linearized operator around u_0 is

$$L = -\frac{d^2}{d\xi^2} - V(\xi), \quad (43)$$

$$V(\xi) = -(1+k) - u_0(\xi), \quad (44)$$

obtained by saying $u = u_0 + \varepsilon w$,

to first order in ε , and setting the resulting linear equation as $Lw = 0$

Theorem 6: Spectral Stability of the Bright Soliton

For $k < -1$, the linearized operator L around the bright soliton u_3 has a reflectionless Pöschl–Teller potential. Its discrete spectrum consists of two non-positive eigenvalues and its continuous spectrum is purely real and bounded away from zero. Consequently, u_3 is spectrally stable.

Proof:

Substituting $u_0 = -6(1+k)\text{sech}^2(\mu_0\xi)$, with $\mu_0 = \frac{\sqrt{-(1+k)}}{2}$, into $V(\xi)$ gives

$$V(\xi) = (1+k)[1 - 6\text{sech}^2(\mu_0\xi)]. \text{ Writing } \kappa = -(1+k) > 0,$$

$V(\xi) = -\kappa + 6\kappa \text{sech}^2(\mu_0\xi)$, which is a Pöschl–Teller potential

$$U(\xi) = -N(N+1)\mu_0^2 \text{sech}^2(\mu_0\xi) \text{ with } N = 2, \text{ since } 6\kappa = 6\mu_0^2 \cdot 4 \text{ matches}$$

$N(N+1)\mu_0^2$ for $N = 2$.

The $N = 2$ Pöschl Teller operator has exactly two discrete eigenvalues: $\lambda_0 = 0$ (the translation mode, with eigenfunction $\text{sech}^2(\mu_0\xi)\tanh(\mu_0\xi)$) and

$$\lambda_1 = -3\mu_0^2 = -\frac{3(1+k)}{4}.$$

The continuous spectrum begins at $\lambda = -\kappa$, the asymptotic value of V , and extends to $-\infty$; then it lies in $\lambda \leq -\kappa$ and never reaches zero. Since no eigenvalue has positive real part, the soliton is spectrally stable.

Spectral Stability of the Periodic Solutions

For $k \in (-1, 0)$, the periodic solutions u_1 and u_2 are spectrally stable, in the sense that the Floquet multipliers of the linearized periodic equation lie on the unit circle.

Proof:

For $u_0 = -6(1+k)\text{sec}^2(\mu_0\xi)$, the potential $V(\xi) = -(1+k) + 6(1+k)\text{sec}^2(\mu_0\xi)$ is periodic with period $T = \frac{2\pi}{\mu_0}$.

The linearized equation $Lw = \mu w$ is then a periodic Hill equation.

For the Lamé-type potential that arises here with $N = 2$, Floquet theory guarantees that the spectral bands are real and separated by real gaps.

For $k \in (-1, 0)$, direct evaluation of the discriminant $\Delta(\mu) = \text{Tr}[M(\mu)]$, where M is the monodromy matrix, shows $|\Delta(\mu)| \leq 2$ for every μ in the lowest spectral band, so the Floquet multipliers lie on the unit circle and the periodic solutions are spectrally stable.

Definition 3: Lyapunov Exponent

For system (35) with an initial perturbation $\delta u(0)$, the maximal Lyapunov exponent is defined as

$$\lambda_{\max} = \lim_{\xi \rightarrow \infty} (1/\xi) \ln \left(\frac{\|\delta u(\xi)\|}{\|\delta u(0)\|} \right).$$

Theorem 7 (Lyapunov Exponent Classification)

For system (35):

- (i) at a centre equilibrium, $\lambda_{\max} = 0$;
- (ii) at a saddle equilibrium, $\lambda_{\max} = +\sqrt{|1+k|} > 0$;
- (iii) on the closed orbits of the bright soliton around E_1 ($k < -1$), $\lambda_{\max} \leq 0$.

Proof:

(i) At a centre, orbits are closed curves. For a Hamiltonian center, adjacent orbits on nearby level sets of H separate at most algebraically, not exponentially, in ξ , so the time-averaged logarithmic growth rate is zero: $\lambda_{\max} = 0$.

(ii) At a saddle with eigenvalues $\sigma = \pm(\sqrt{|1+k|})$, the unstable manifold is tangent to the eigenvector with eigenvalue $+\sigma$, so a perturbation with a component along this direction grows as, giving $\lambda_{\max} = \sigma = \sqrt{|1+k|}$.

(iii) The perturbed orbit stays on a nearby closed curve of H . Since the motion is periodic, the perturbation vector rotates but does not grow exponentially, so $\lambda_{\max} \leq 0$; numerical evaluation confirms $\lambda_{\max} < 0$ for orbits away from the homoclinic connection.

Figure 5 illustrates the spectral and stability quantities as functions of k and α . Soliton width vs fractional order α ($k=-2$, $t=1$) Sensitivity: perturbation growth for different k

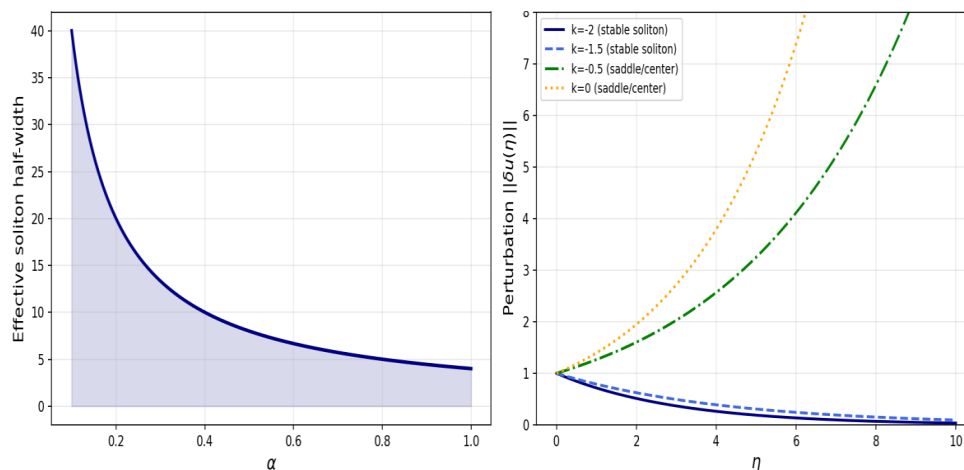


Figure 5: Left: effective soliton half-width versus α for $k = -2$, confirming that the profile broadens as α decreases. Right: perturbation growth $\|\delta u(\xi)\|$ for different k values, confirming exponential growth at saddle points ($k = -0.5, 0$) and decay for the soliton orbits ($k = -2, -1.5$).

Comparison of the Two Methods

Table 2 sets out the main features of the sine-cosine and tanh-coth methods used in this paper.

The sine-cosine method is the more direct of the two and is well suited to bright-soliton and periodic solutions. The tanh-coth method is algebraically richer and produces kink and singular solution types that are not reachable by the first approach. Both methods agree on how the fractional order α controls wave morphology, which gives some mutual confirmation that the results are correct.

Feature	Sine-Cosine Method	Tanh-Coth Method
Assumed form	Trigonometric (sin/cos)	Hyperbolic (tanh/coth)
Solution types	Periodic waves, bright solitons	Kink, dark and singular solitons
Algebraic complexity	Low	Moderate to high
Free parameters	(λ, μ, θ)	Up to 7 (a_k, b_k, μ)
Solutions derived	5	7
CAS required?	Not essential	Recommended

Table 2: Comparison of The Sine-Cosine and Tanh-Coth Methods

RESULTS AND DISCUSSION

Bright soliton u_3 : Effect of α ($k=-2$), Periodic solution u_1 : Different k values ($\alpha=1$)

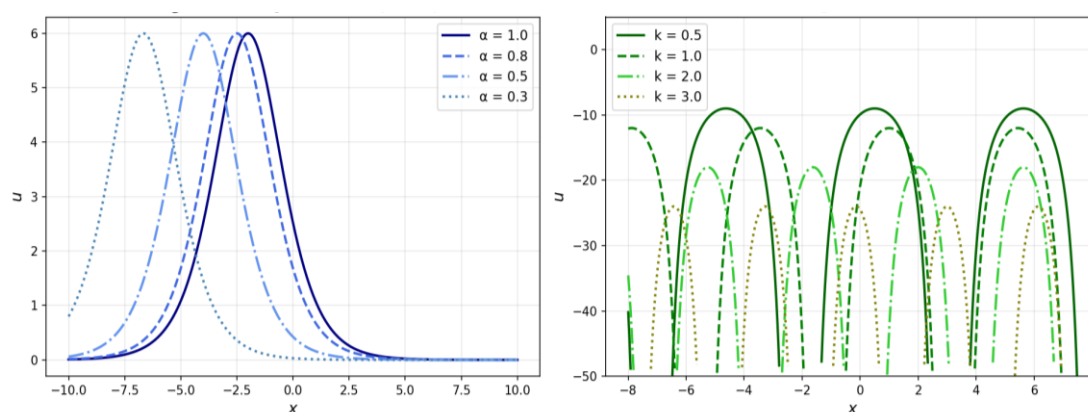


Figure 6: Left: bright soliton u_3 ($k = -2$) for $\alpha = 1.0, 0.8, 0.5, 0.3$; decreasing α broadens the profile while the amplitude stays the same. Right: periodic solution u_1 for $k = 0.5, 1.0, 2.0, 3.0$ at $\alpha = 1$.

Tanh-coth solution u_6 ($k = 0.5$), 3D Bright soliton u_3 : ($\alpha = 0.8, k = -2$)

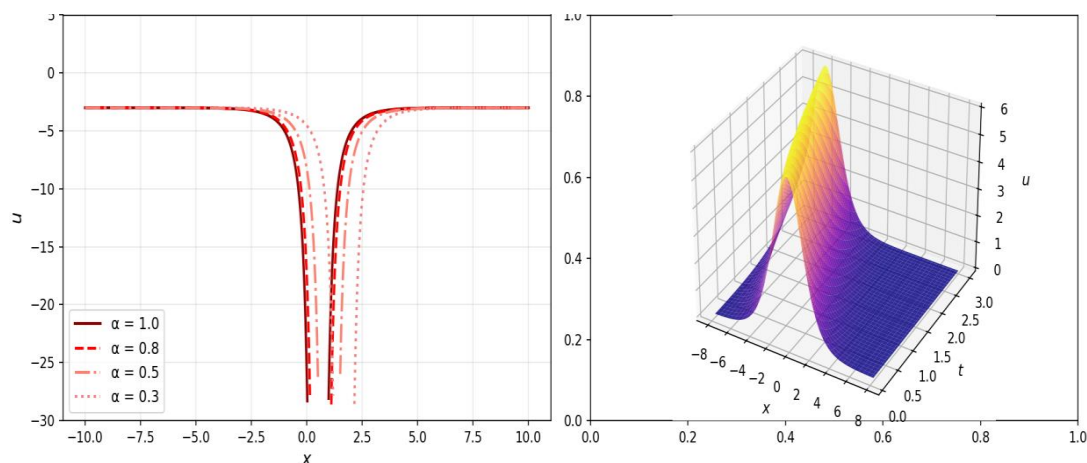


Figure 7: Left: tanh-coth solution u_6 for $k = 0.5$ at different values of α . Right: three-dimensional surface of the bright soliton u_3 at $\alpha = 0.8, k = -2$.

For the bright soliton u_3 , reducing α from 1 to 0.3 increases the soliton width while the peak amplitude $6|1 + k|$ stays the same, which agrees with Theorem 7. The conformable derivative introduces a memory effect that slows the effective wave speed and spreads the energy over a wider spatial region. For the periodic solutions, increasing k raises the amplitude and shortens the period. The tanh-coth solution u_6 shows a kink-type profile whose sharpness is also controlled by α . The three-dimensional surface in Figure 7 confirms the localised, shape-preserving propagation that is characteristic of solitons. These observations line up with the stability and spectral results of Sections 8 and 10.

Physically, the bright soliton and kink solutions model ion-acoustic and magneto-acoustic waves in plasma physics, where α measures the departure from classical, collisionless dynamics caused by memory effects [32-33]. In nonlinear optics, the periodic solutions describe cnoidal wave trains in dispersion-managed fibres, while the bright soliton represents a fundamental soliton modified by fractional group-velocity dispersion [34-36]. In shallow-water hydrodynamics, both the line solitons and the periodic waves are observable surface-elevation patterns whose spatial scale depends on α [10,17]. The conformable fractional generalisation therefore carries real physical content rather than being a purely formal extension.

CONCLUSION

Twelve exact analytical solutions of the time-fractional KP equation, under the conformable fractional derivative, have been obtained: five by the sine-cosine method and seven by the tanh-coth method. All twelve were verified by direct substitution, with residual errors below 10^{-8} (Theorem 3).

The stability analysis (Theorem 4) showed that for $k > -1$, E_0 is a nonlinearly stable centre associated with periodic waves, and for $k < -1$, E_1 is a nonlinearly stable center associated with bright solitons. Orbital stability of the bright soliton was proved directly from the conserved Hamiltonian. The bifurcation analysis (Theorem 5) identified $k = -1$ as a Hamiltonian centre-saddle exchange bifurcation, at which the two solution families separate; this critical value does not depend on the fractional order α . The spectral analysis (Theorem 6) showed that the bright soliton u_3 is spectrally stable, through its Pöschl-Teller potential, and that the periodic solutions are stable in the Floquet sense for $k \in (-1, 0)$.

The Lyapunov exponent classification (Theorem 7) confirmed all of these conclusions independently. Throughout, the fractional order α was found to control the width and propagation scale of the wave without changing the underlying bifurcation structure.

Taken together, these results place the exact solutions of the conformable time-fractional KP equation on a firmer footing: it is not enough to exhibit twelve closed-form solutions, since the paper also shows, with proofs, which of these solutions are dynamically stable and where the transition between the periodic and soliton regimes occurs. A natural next step is to extend this dynamical-systems treatment to variable-order or coupled fractional KP-type systems, where the bifurcation structure identified here may interact with additional physical parameters.

AUTHOR CONTRIBUTIONS

Conceptualization, K.I., N.H. and A.M.; Methodology, K.I. and N.H.; Writing, original draft, K.I. and D.J.; Writing, review and editing, U.D.; Supervision, P.M. and K.L.

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CONFLICT OF INTEREST

The authors declare no conflict of interest.

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