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Exploring Analytical Approaches for Fractional Order Zakharov–Kuznetsov Equation

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ABSTRACT

This study applies the Natural Transform Iterative Method (NTIM) and Homotopy Perturbation Method (HPM) to approximate solutions for fractional order two-dimensional partial differential equations, focusing on the Zakharov–Kuznetsov equation. It uses Caputo fractional derivatives and Mathematica for efficient computation, producing rapidly convergent series solutions. Rigorous comparisons with existing methods affirm the method's robustness and efficiency in solving higher-dimensional fractional order equations, demonstrating its computational efficacy and practical utility in addressing complex mathematical challenges posed by fractional order systems.

Keywords: Homotopy Perturbation Method: Caputo Derivative; Natural Transform Iterative Method; Natural Transform; Analytical Techniques

INTRODUCTION

Fractional calculus (FC), which originated in the era of Newton, has recently garnered significant scholarly interest. Over the pastthree decades, notable advancements in scientific and engineering applications have emerged within the domain of FC. Thefractional derivative concept has evolved to address the intricacies associated with heterogeneous phenomena. Fractionaldifferential operators prove effective in capturing the intricate behavior of complex media undergoing diffusion processes.Serving as an indispensable tool, differential equations of arbitrary order facilitate a more convenient and accurate illustration ofnumerous problems. The rapid progress in mathematical techniques, coupled with advanced computer software, has promptedresearchers to delve into generalized calculus, enabling them to articulate their perspectives when analyzing intricate phenomena.

Fractional calculus, an extension of integer-order calculus, was historically perceived as a purely mathematical discipline with no tangible applications in the real world. However, recent advancements have disproven this notion, showcasing the utility of fractional calculus in various practical scenarios. Applications include modeling sound wave propagation in rigid porous materials, analyzing ultra-sonic wave propagation in human cancellous bone, characterizing the viscoelastic properties of soft biological tissues, and addressing the path tracking problem in autonomous electric vehicles [1-4].

The focal point of numerous studies lies in differential equations of fractional order, given their frequent utilization in diverse fields such as electromagnetic phenomena, electrochemistry, acoustics, material science, physics, viscoelasticity, and engineering [5-9]. These problems are notably more intricate compared to integer-order differential equations. Due to the inherent complexities of fractional calcu-lus, a significant portion of fractional order differential equations lacks exact solutions. Consequently, researchers extensively employ approximate methods for solving these equations [10-14]. Recent methods for approximating solutions to fractional order differential equations include semi-analytical techniques [15-26].

This study extends the NTIM and HPM formulations to fractional order partial differential equa-tions in two dimensions. In particular, examples of the fractional version of the Zakharov–Kuznetsov equations, or F ZK(˜p, q˜, r˜), are used to demonstrate the extended formulation.

$$\tilde{V}(\gamma,\xi,\mu) = D_{\mu}^{\beta}\tilde{V} + \breve{a}(\tilde{V}^{\tilde{p}})_{\gamma} + \breve{b}(\tilde{V}^{\tilde{q}})_{\gamma,\gamma,\gamma} + \breve{c}(\tilde{V}^{\tilde{r}})_{\gamma,\xi,\xi} = 0. \tag{1}$$

The provided equation involves the parameter β, which characterizes the fractional derivative theory with a restriction of (0 < β ≤ 1). The constants ˇa, bˇ and cˇ are arbitrary, whereas p˜, q˜, r˜ = 0 are non-zero integers that affect the behavior of weakly nonlinear ion-acoustic waves in a plasma with cold ions and hot isothermal electrons under a uniform magnetic field. Numerous researchers have successfully addressed the the FZK equation, employing various methodologies, as outlined in recent studies [27-31]. These diverse approaches highlight the breadth of techniques applied to solve this equation, underscoring the significance of understanding its solutions in the context of plasma physics. There are six sections in this study. Basic concepts and characteristics from fractional calculus are given in Section 2. The study of NTIM and HPM for two-dimensional FPDEs is the main topic of Section 3. The FZK (2, 2, 2) and FZK (3, 3, 3) equation's second-order approximation solutions are shown in Section 4, where the temporal fractional derivatives are specified in the Caputo sense. In Section 5, the second-order approximation solutions produced by the suggested techniques are con-trasted with the results of the Perturbation-Iteration Algorithm (PIA) and the third-order Variational Iteration Method (VIM). The suggested approaches show better outcomes in every case.

Basic Definitions

Here, we explore two key ideas that are fundamental to the present framework: the Fundamentals of Fractional Calculus and the Natural Transform [32].

Definition

Using the Riemann-Liouville technique $\tilde{\varpi}(\tilde{\kappa},\tilde{\mathrm{T}})\,\epsilon\, C_\eta(\eta \geq -1)$, the fractional integral for a function

$$J^{\phi}\tilde{\varpi}(\tilde{\kappa},\tilde{\mathrm{T}}) = \frac{1}{\Gamma(\phi)}\int_0^{\tilde{\mathrm{T}}}(\tilde{\mathrm{T}}-\psi)^{\phi-1}\mathrm{T}\varpi(\tilde{\kappa},\psi)d\psi$$

$$J^{0}\tilde{\varpi}(\tilde{\kappa},\tilde{\mathrm{T}}) = \tilde{\varpi}(\tilde{\kappa},\tilde{\mathrm{T}}).$$

Definition

According to the Caputo method, the fractional-order derivative is as follows

$$D^{\phi}_{\tilde{\mathrm{T}}}\varpi(\tilde{\kappa},\tilde{\mathrm{T}}) = \frac{1}{\Gamma(\bar{n}-\phi)}\int_0^{\tilde{\mathrm{T}}}(\tilde{\mathrm{T}}-\psi)^{\bar{n}-\phi-1}\varpi^{\bar{n}}(\tilde{\kappa},\psi)d\psi, \qquad \bar{n}-1 \;\leq\; \phi \;\leq\; \bar{n},\; n\,\epsilon\, N. \tag{2}$$

Definition

The natural transform of ψ(t) is defined as

$$N^{+}\Big[\psi(t)\Big] = R(\delta,u) = \frac{1}{u}\int_0^{\infty} e^{\frac{-\delta t}{u}}(\psi(t))dt. \tag{3}$$

Where $\delta, u > 0$ are the transform variables.

Definition

If R(δ, u) is ψ(t) natural transform, then ψ(t) inverse natural transform is described as follows:

$$N^{-}\Big[R(\delta,u)\Big] = \psi(t) = \frac{1}{2\pi\iota}\int_{c-\iota\infty}^{c+\iota\infty} e^{\frac{\delta t}{u}}(R(\delta,u))d\delta. \tag{4}$$

The integral is represented as δ = a + bι in the complex plane along δ = c, where c ϵ R.

Definition

Given is the nth derivative of the natural transform of ψ(t)

$$N^{+}\Big[\psi^{n}(t)\Big] = R_n(\delta,u) = \frac{\delta^n}{u^n}R(\delta,u) - \Sigma_{k=0}^{n-1}\frac{\delta^{n-k-1}}{u^{n-k}}\Big[\psi^{n}(0)\Big], \quad n \;\geq\; 1. \tag{5}$$

Basic Procedure of Methods

Natural Transform Iterative Method

Take into consideration FDE of the following form [33]:

$$D_{\tilde{\sigma}}^{\eta}\Big(\bar{\Theta}(\tilde{\gamma},\tilde{\sigma})\Big)=f(\tilde{\gamma},\tilde{\sigma})+\tilde{\varpi}\Big(\bar{\Theta}(\tilde{\gamma},\tilde{\sigma})\Big)+\tilde{\psi}\Big(\bar{\Theta}(\tilde{\gamma},\tilde{\sigma})\Big),\ \tilde{\gamma},\tilde{\sigma}\ \geq\ 0,\tilde{m}-1\ \leq\ \beta\ \leq\ \tilde{m}. \tag{6}$$

where the caputo fractional derivative of order $\eta,\tilde{m}\ \epsilon\ \mathbf{N}$ and $(\gamma=y_1,y_2,\cdots,y_{\tilde{m}})$ are represented by $D_{\tilde{\sigma}}^{\eta}$. $\tilde{\varpi}$ and $\tilde{\psi}$, respectively, stand for the linear and non-linear functions. The function $f(\tilde{\gamma},\tilde{\sigma})$ is

known. The corresponding initial condition are given as follows

$$\bar{\Theta}(\tilde{\gamma},0)=\tilde{\varphi}(\tilde{\gamma}). \tag{7}$$

Using the natural transform, equation (6) has undergone a natural transformation.

$$\aleph^{+}\Big[D_{\tilde{\sigma}}^{\eta}\Big[\bar{\Theta}\Big(\tilde{\gamma},\tilde{\sigma}\Big)\Big]\Big]=\aleph^{+}\Big[f(\tilde{\gamma},\tilde{\sigma})\Big]+\aleph^{+}\Big[\varpi\Big(\bar{\Theta}(\tilde{\gamma},\tilde{\sigma})\Big)+\tilde{\psi}\Big(\bar{\Theta}(\tilde{\gamma},\tilde{\sigma})\Big)\Big]. \tag{8}$$

Using the definition, Equation (8) can be represented as

$$\frac{s^{\eta}}{\bar{\Theta}^{\eta}}\aleph^{+}\Big[\bar{\Theta}\Big(\tilde{\gamma},\tilde{\sigma}\Big)\Big]-\frac{s^{\eta-1}}{\bar{\Theta}^{\eta}}\bar{\Theta}(\tilde{\gamma},0)=\aleph^{+}\Big[f(\tilde{\gamma},\tilde{\sigma})\Big]+\aleph^{+}\Big[\varpi\Big(\bar{\Theta}(\tilde{\gamma},\tilde{\sigma})\Big)+\tilde{\psi}\Big(\bar{\Theta}(\tilde{\gamma},\tilde{\sigma})\Big)\Big]. \tag{9}$$

When we arrange equation (9), we obtain

$$\aleph^{+}\Big[\bar{\Theta}\Big(\tilde{\gamma},\tilde{\sigma}\Big)\Big]=\frac{\varphi(\tilde{\gamma})}{s}+\frac{\bar{\Theta}^{\eta}}{s^{\eta}}\left(\aleph^{+}[f(\tilde{\gamma},\tilde{\sigma})]\right)+\frac{\bar{\Theta}^{\eta}}{s^{\eta}}\left(\aleph^{+}\Big[\varpi\Big(\bar{\Theta}(\tilde{\gamma},\tilde{\sigma})\Big)+\tilde{\psi}\Big(\bar{\Theta}(\tilde{\gamma},\tilde{\sigma})\Big)\Big]\right). \tag{10}$$

When computing the NTIM solution, $\bar{\Theta}(\tilde{\gamma},\tilde{\sigma})$ is extended as

$$\bar{\Theta}(\tilde{\gamma},\tilde{\sigma})=\sum_{\tilde{i}=0}^{\infty}\bar{\Theta}_{i}(\tilde{\gamma},\tilde{\sigma}). \tag{11}$$

and the non-linear term $\tilde{\psi}\Big(\bar{\Theta}(\tilde{\gamma},\tilde{\sigma})\Big)$ is defined as

$$\tilde{\psi}\Big(\sum_{\tilde{m}=0}^{\infty}\bar{\Theta}_{\tilde{m}}(\tilde{\gamma},\tilde{\sigma})\Big)=\tilde{\psi}\Big(\bar{\Theta}_0(\tilde{\gamma},\tilde{\sigma})\Big)+\sum_{\tilde{m}=1}^{\infty}\Big\{\tilde{\psi}\Big(\sum_{\tilde{j}=0}^{i}\bar{\Theta}_{\tilde{j}}(\tilde{\gamma},\tilde{\sigma})\Big)-\tilde{\psi}\Big(\sum_{\tilde{j}=0}^{\tilde{j}-1}\bar{\Theta}_{\tilde{j}}(\tilde{\gamma},\tilde{\sigma})\Big)\Big\}. \tag{12}$$

Using Equation (11) and Equation (12) in Equation (10), we obtain

$$\aleph^{+}\Big[\sum_{\tilde{i}=1}^{\infty}\bar{\Theta}_{\tilde{i}}\Big]=\frac{\varphi(\tilde{\gamma})}{s}+\frac{\bar{\Theta}^{\eta}}{s^{\eta}}\Big(\aleph^{+}[f(\tilde{\gamma},\tilde{\sigma})]\Big)+\frac{\bar{\Theta}^{\eta}}{s^{\eta}}\Big[\aleph^{+}\Big[\sum_{\tilde{m}=0}^{\infty}\varpi\Big(\bar{\Theta}_{\tilde{m}}\Big)+\tilde{\psi}(\bar{\Theta}_0)+\sum_{\tilde{m}=1}^{\infty}\Big\{\tilde{\psi}(\sum_{\tilde{j}=0}^{\tilde{m}}\bar{\Theta}_{\tilde{j}})-\tilde{\psi}(\sum_{\tilde{j}=0}^{\tilde{m}-1}\bar{\Theta}_{\tilde{j}})\Big\}\Big]\Big]. \tag{13}$$

Utilising the recursive relationship

$$\aleph^{+}\Big[\bar{\Theta}_0(\tilde{\gamma},\tilde{\sigma})\Big]=\frac{\varphi(\tilde{\gamma})}{s}+\frac{\bar{\Theta}^{\eta}}{s^{\eta}}\Big(\aleph^{+}[f(\tilde{\gamma},\tilde{\sigma})]\Big),$$

$$\aleph^{+}\Big[\bar{\Theta}_1(\tilde{\gamma},\tilde{\sigma})\Big]=\frac{\bar{\Theta}^{\eta}}{s^{\eta}}\Big[\varpi(\bar{\Theta}_0(\tilde{\gamma},\tilde{\sigma}))+\tilde{\psi}(\bar{\Theta}_0(\tilde{\gamma},\tilde{\sigma}))\Big],$$

$$\aleph^{+}\Big[\bar{\Theta}_2(\tilde{\gamma},\tilde{\sigma})\Big]=\frac{\bar{\Theta}^{\eta}}{s^{\eta}}\Big[\varpi(\bar{\Theta}_1(\tilde{\gamma},\tilde{\sigma}))+\tilde{\psi}(\bar{\Theta}_0(\tilde{\gamma},\tilde{\sigma})+\bar{\Theta}_1(\tilde{\gamma},\tilde{\sigma}))-\tilde{\psi}(\bar{\Theta}_0(\tilde{\gamma},\tilde{\sigma}))\Big],$$

$$\vdots$$

$$\aleph^{+}\left[\bar{\Theta}_{\tilde{i}+1}(\tilde{\gamma},\tilde{\sigma})\right] = \frac{\bar{\Theta}^{\eta}}{s^{\eta}}\left[\varpi(\bar{\Theta}_{\tilde{i}}(\tilde{\gamma},\tilde{\sigma})) + \tilde{\psi}(\bar{\Theta}_0(\tilde{\gamma},\tilde{\sigma}) + \bar{\Theta}_1(\tilde{\gamma},\tilde{\sigma}) + \cdots + \bar{\Theta}_{\tilde{i}}(\tilde{\gamma},\tilde{\sigma})) - \tilde{\psi}(\bar{\Theta}_0(\tilde{\gamma},\tilde{\sigma}) + \bar{\Theta}_1(\tilde{\gamma},\tilde{\sigma}) + \cdots + \bar{\Theta}_{\tilde{i}}(\tilde{\gamma},\tilde{\sigma}))\right], \tilde{i} \geq 0. \tag{14}$$

Using equation (14) as an inverse natural transform, we get

$$\bar{\Theta}_0(\tilde{\gamma},\tilde{\sigma}) = \aleph^{-}\left[\frac{\varphi(\tilde{\gamma})}{s} + \frac{\bar{\Theta}^{\eta}}{s^{\eta}}\left(\aleph^{+}[f(\tilde{\gamma},\tilde{\sigma})]\right)\right]$$

$$\bar{\Theta}_1(\tilde{\gamma},\tilde{\sigma}) = \aleph^{-}\left[\frac{\bar{\Theta}^{\eta}}{s^{\eta}}\left[\varpi(\bar{\Theta}_0(\tilde{\gamma},\tilde{\sigma})) + \tilde{\psi}(\bar{\Theta}_0(\tilde{\gamma},\tilde{\sigma}))\right]\right],$$

$$\bar{\Theta}_1(\tilde{\gamma},\tilde{\sigma}) = \aleph^{-}\left[\frac{\bar{\Theta}^{\eta}}{s^{\eta}}\left[\varpi(\bar{\Theta}_1(\tilde{\gamma},\tilde{\sigma})) + \tilde{\psi}(\bar{\Theta}_0(\tilde{\gamma},\tilde{\sigma}) + \bar{\Theta}_1(\tilde{\gamma},\tilde{\sigma})) - \tilde{\psi}(\bar{\Theta}_0(\tilde{\gamma},\tilde{\sigma}))\right]\right],$$

$$\vdots$$

$$\bar{\Theta}_{\tilde{i}+1}(\tilde{\gamma},\tilde{\sigma}) = \aleph^{-}\left[\frac{\bar{\Theta}^{\eta}}{s^{\eta}}\left[\varpi(\bar{\Theta}_{\tilde{i}}(\tilde{\gamma},\tilde{\sigma})) + \tilde{\psi}(\bar{\Theta}_0(\tilde{\gamma},\tilde{\sigma}) + \bar{\Theta}_1(\tilde{\gamma},\tilde{\sigma}) + \cdots + \bar{\Theta}_{\tilde{i}}(\tilde{\gamma},\tilde{\sigma})) - \tilde{\psi}(\bar{\Theta}_0(\tilde{\gamma},\tilde{\sigma}) + \bar{\Theta}_1(\tilde{\gamma},\tilde{\sigma}) + \cdots + \bar{\Theta}_{\tilde{i}}(\tilde{\gamma},\tilde{\sigma}))\right]\right], \tilde{i} \geq 0. \tag{15}$$

After adding up all the components, the approximate solutions to equations (6) and (8) using NTIM are given as follows

$$\bar{\Theta}(\tilde{\gamma},\tilde{\sigma}) = \bar{\Theta}_0(\tilde{\gamma},\tilde{\sigma}) + \bar{\Theta}_1(\tilde{\gamma},\tilde{\sigma}) + \cdots + \bar{\Theta}_{\tilde{m}-1}(\tilde{\gamma},\tilde{\sigma}), \tilde{m} \,\epsilon\, N. \tag{16}$$

Bhalekar and Daftardar-Gejji assert that NTIM convergence is equal to NIM convergence [34].

Homotopy Perturbation Method

Take the following nonlinear differential equation to illustrate the operation of HPM:

$$\varphi[\mathfrak{L}(\mathfrak{z})] + \tilde{P}(\mathfrak{u}) = 0, \qquad \mathfrak{u} \,\epsilon\, \Psi, \tag{17}$$

with the boundary conditions

$$\breve{\beta}\left(\mathfrak{L}(\mathfrak{z}), \frac{\partial \mathfrak{L}(\mathfrak{z})}{\mathfrak{z}}\right) = 0, \qquad \mathfrak{u} \,\epsilon\, \Lambda. \tag{18}$$

φ is a basic differential operator, $\breve{\beta}$ is a boundary operator, $\tilde{P}(\mathfrak{u})$ is a well known analytic function, and Λ is the domain boundary for Ψ.

The operator φ is decomposed into two pieces, χ and ξ, where χ is linear and ξ is nonlinear. Hence, Equation (17) may be expressed as follows:

$$\chi[\mathfrak{L}(\mathfrak{z})] + \xi[\mathfrak{L}(\mathfrak{z})] - \tilde{P}(\mathfrak{u}) = 0, \tag{19}$$

He [35] built a homotopy $\mathfrak{L} : \Psi \times [0,1] \rightarrow \mathfrak{R}$ that meets the condition,

$$\mathcal{H}(\mathfrak{L},\rho) = (1-\rho)[\chi(\mathfrak{L}) - \chi(\mathfrak{L}_0)] + \rho[\varphi(\mathfrak{L}) - \tilde{P}(\mathfrak{u})] \tag{20}$$

or

$$\mathcal{H}(\mathfrak{L},\rho) = \chi(\mathfrak{L}) - \chi(\mathfrak{L}_0) + \rho[\chi(\mathfrak{L}_0)] + \rho[\xi(\mathfrak{L}) - \tilde{P}(\mathfrak{u})]. \tag{21}$$

where $\mathfrak{u} \,\epsilon\, \Psi$, $\rho \,\epsilon\, [0,1]$ which is referred to as the homotopy parameter, and $\mathfrak{L}_0$ is the initial approxi-mation of the function (17). Hence, it concludes that

$$\mathcal{H}(\mathfrak{L},0) = \chi(\mathfrak{L}) - \chi(\mathfrak{L}_0) = 0, \qquad \mathcal{H}(\mathfrak{L},1) = \varphi(\mathfrak{L}) - \tilde{P}(\mathfrak{u}) = 0 \tag{22}$$

and the procedure of moving ρ from 0 to 1 is identical to that of $\mathcal{H}(\mathfrak{L}, \rho)$ from $\chi(\mathfrak{L}) - \chi(\mathfrak{L}_0)$ to $\varphi(\mathfrak{L}) - \tilde{P}(\mathfrak{u})$. This is known as deformation in topology, $\chi(\mathfrak{L}) - \chi(\mathfrak{L}_0)$ and $\varphi(\mathfrak{L}) - \tilde{P}(\mathfrak{u})$ are called homotopic. Using the perturbation approach, and assuming that $0 \leq \rho \leq 1$ is a small parameter, we may suppose that the solution of (20) or (21) can be written as a series in ρ, as shown below

$$\mathfrak{L} = \mathfrak{L}_0 + \rho\mathfrak{L}_1 + \rho^2\mathfrak{L}_2 + \rho^3\mathfrak{L}_3 + \cdots \tag{23}$$

when $\rho \to 1$,(20) or (21) corresponds to (19) and becomes the approximate solution of (19), i.e,

$$\mathfrak{L}(\mathfrak{z}) = \lim_{\rho \to 1} \mathfrak{L} = \mathfrak{L}_0 + \mathfrak{L}_1 + \mathfrak{L}_2 + \mathfrak{L}_3 + \cdots \tag{24}$$

The convergence rate of the series (24) is dependent on φ(L) in the majority of situations.

Solving Time Fractional FZK (2, 2, 2)

Examine the following equation and its accompanying initial conditions.

$$\frac{\partial^\beta V}{\partial \mu^\alpha} + \frac{\partial^2 V^2}{\partial \gamma^2} + \frac{1}{8}\frac{\partial^3 V^2}{\partial \gamma^3} + \frac{1}{8}\frac{\partial^3 V^2}{\partial \gamma \partial \xi^2} = 0, \qquad 0 < \alpha \leq 1. \tag{25}$$

$$V(\gamma, \xi, 0) = \frac{4}{3}\lambda \sinh^2(\gamma + \xi).$$

A precise solution to equation (25) for $\beta = 1$,

$$V(\gamma, \xi, \mu) = \frac{4}{3}\lambda \sinh^2(\gamma + \xi - \lambda\mu). \tag{26}$$

where λ is an arbitrary constant.

Application of NTIM

Applying natural transform on Equation (25), we get

$$N\Big(\frac{\partial^\beta V}{\partial \mu^\beta}\Big) = N\Big(-\frac{\partial^2 V^2}{\partial \gamma^2} - \frac{1}{8}\frac{\partial^3 V^2}{\partial \gamma^3} - \frac{1}{8}\frac{\partial^3 V^2}{\partial \gamma \partial \xi^2}\Big). \tag{27}$$

By applying the differentiation property of the natural transform to Equation (27), we obtain the following result

$$\frac{s^\beta}{\nu^\beta}V(\gamma, \xi, \mu) - \frac{\nu^{\beta-1}}{s^\beta}V(\gamma, \xi, 0) = N\Big(-\frac{\partial^2 V^2}{\partial \gamma^2} - \frac{1}{8}\frac{\partial^3 V^2}{\partial \gamma^3} - \frac{1}{8}\frac{\partial^3 V^2}{\partial \gamma \partial \xi^2}\Big). \tag{28}$$

By applying the inverse natural transform to Equation (28), we obtain

$$V(\gamma, \xi, \mu) = \frac{V(\gamma, \xi, 0)}{s} + N^-\Big\{\frac{\nu^\beta}{s^\beta}N\Big(-\frac{\partial^2 V^2}{\partial \gamma^2} - \frac{1}{8}\frac{\partial^3 V^2}{\partial \gamma^3} - \frac{1}{8}\frac{\partial^3 V^2}{\partial \gamma \partial \xi^2}\Big). \tag{29}$$

Utilizing the recursive relation from Equation (15),

$$V_0(\gamma, \xi, \mu) = N^-\Big\{\frac{V(\gamma, \xi, 0)}{s}\Big\}.$$

$$V_1(\gamma, \xi, \mu) = N^-\Big\{\frac{\nu^\beta}{s^\beta}N\Big(-\frac{\partial^2 V_0^2}{\partial \gamma^2} - \frac{1}{8}\frac{\partial^3 V_0^2}{\partial \gamma^3} - \frac{1}{8}\frac{\partial^3 V_0^2}{\partial \gamma \partial \xi^2}\Big)\Big\}.$$

$$\begin{aligned} V_2(\gamma, \xi, \mu) =& N^-\Big\{\frac{\nu^\beta}{s^\beta}N\Big\{\Big(-\frac{\partial^2 (V_0+V_1)^2}{\partial \gamma^2} - \frac{1}{8}\frac{\partial^3 (V_0+V_1)^2}{\partial \gamma^3} - \frac{1}{8}\frac{\partial^3 (V_0+V_1)^2}{\partial \gamma \partial \xi^2}\Big) \\ &- \Big(-\frac{\partial^2 V_0^2}{\partial \gamma^2} - \frac{1}{8}\frac{\partial^3 V_0^2}{\partial \gamma^3} - \frac{1}{8}\frac{\partial^3 V_0^2}{\partial \gamma \partial \xi^2}\Big)\Big\}. \end{aligned}$$

By using the software Wolfram Mathematica 13.2, the solution components are obtained as

$$V_0(\gamma, \xi, \mu) = \frac{4}{3}\lambda \sin^2(\gamma + \xi). \tag{30}$$

$$V_1(\gamma,\xi,\mu) = \frac{16\lambda^2\mu^\beta(\sin(2(\gamma+\xi)) - 2\sin(4(\gamma+\xi)) - 2\cos(2(\gamma+\xi)) + 2\cos(4(\gamma+\xi)))}{9\Gamma(\beta+1)}. \quad (31)$$

$$\begin{aligned} V_2(\gamma,\xi,\mu) =& \frac{128\lambda^3\mu^{2\beta}}{81\Gamma(2\beta+1)}\Big(3(-4\cos(2(\gamma+\xi)) + 4\cos(4(\gamma+\xi)) + 9\cos(6(\gamma+\xi)) \\ &+ 7\sin(2(\gamma+\xi)) - 44\sin(4(\gamma+\xi)) + 45\sin(6(\gamma+\xi))) - \frac{8\lambda\Gamma(2\beta+1)^2\mu^\beta}{\Gamma(\beta+1)^2\Gamma(3\beta+1)} \\ &(5\cos(2(\gamma+\xi)) + 2\cos(4(\gamma+\xi)) - 63\cos(6(\gamma+\xi)) + 128\cos(8(\gamma+\xi)) \\ &- 5\sin(2(\gamma+\xi)) + 14\sin(4(\gamma+\xi)) - 81\sin(6(\gamma+\xi)) + 64\sin(8(\gamma+\xi)))\Big). \end{aligned} \quad (32)$$

By combining the components, the second-order approximate NTIM solution is expressed as.

$$V_{NTIM}(\gamma,\xi,\mu) = V_0(\gamma,\xi,\mu) + V_1(\gamma,\xi,\mu) + V_2(\gamma,\xi,\mu). \quad (33)$$

Application of HPM

Applying the HPM formulation outlined in Section 3, we obtain

Zero-order component:

$$\frac{\partial^\beta V_0}{\partial\mu^\beta} = 0,$$

$$V(\gamma,\xi,0) = \frac{4}{3}\lambda\sin^2(\gamma+\xi). \quad (34)$$

First-order component:

$$\begin{aligned} \frac{\partial^\beta V_1}{\partial\mu^\beta} =& -\left(\frac{1}{4}V_0\frac{\partial^3 V_0}{\partial\gamma\,\partial\xi^2} + \frac{1}{4}\frac{\partial V_0}{\partial\gamma}\frac{\partial^2 V_0}{\partial\xi^2} + \frac{1}{2}\frac{\partial V_0}{\partial\xi}\frac{\partial^2 V_0}{\partial\gamma\,\partial\xi} + \frac{1}{4}V_0\frac{\partial^3 V_0}{\partial\gamma^3} + \frac{3}{4}\frac{\partial V_0}{\partial\gamma}\frac{\partial^2 V_0}{\partial\gamma^2}\right. \\ &\left.+ 2V_0\frac{\partial^2 V_0}{\partial\gamma^2} + 2\left(\frac{\partial V_0}{\partial\gamma}\right)^2\right). \end{aligned}$$

Second-order component:

$$\begin{aligned} \frac{\partial^\beta V_2}{\partial\mu^\beta} =& -\left(\frac{1}{4}V_1\frac{\partial^3 V_0}{\partial\gamma\,\partial\xi^2} + \frac{1}{4}V_0\frac{\partial^3 V_1}{\partial\gamma\,\partial\xi^2} + \frac{1}{4}\frac{\partial V_1}{\partial\gamma}\frac{\partial^2 V_0}{\partial\xi^2} + \frac{1}{2}\frac{\partial V_1}{\partial\xi}\frac{\partial^2 V_0}{\partial\gamma\,\partial\xi} + \frac{1}{2}\frac{\partial V_0}{\partial\xi}\frac{\partial^2 V_1}{\partial\gamma\,\partial\xi}\right. \\ &+ \frac{1}{4}\frac{\partial V_0}{\partial\gamma}\frac{\partial^2 V_1}{\partial\xi^2} + \frac{1}{4}V_1\frac{\partial^3 V_0}{\partial\gamma^3} + \frac{1}{4}V_0\frac{\partial^3 V_1}{\partial\gamma^3} + \frac{3}{4}\frac{\partial V_1}{\partial\gamma}\frac{\partial^2 V_0}{\partial\gamma^2} + 2V_1\frac{\partial^2 V_0}{\partial\gamma^2} + 2V_0\frac{\partial^2 V_1}{\partial\gamma^2} \\ &\left.+ \frac{3}{4}\frac{\partial V_0}{\partial\gamma}\frac{\partial^2 V_1}{\partial\gamma^2} + 4\frac{\partial V_0}{\partial\gamma}\frac{\partial V_1}{\partial\gamma}\right). \end{aligned}$$

The solutions for the aforementioned components are as follows:

$$V_0(\gamma,\xi,\mu) = \frac{4}{3}\lambda\sin^2(\gamma+\xi). \quad (35)$$

$$V_1(\gamma,\xi,\mu) = \frac{16\lambda^2\mu^\beta(\sin(2(\gamma+\xi)) - 2\sin(4(\gamma+\xi)) - 2\cos(2(\gamma+\xi)) + 2\cos(4(\gamma+\xi)))}{9\beta\Gamma(\beta)}. \quad (36)$$

$$\begin{aligned} V_2(\gamma,\xi,\mu) =& \frac{1}{27\beta^2\Gamma(\beta)^2}\Big(128\lambda^3\mu^{2\beta}(7\sin(2(\gamma+\xi)) - 44\sin(4(\gamma+\xi)) + 45\sin(6(\gamma+\xi)) - 4\cos(2(\gamma+\xi)) \\ &+ 4\cos(4(\gamma+\xi)) + 9\cos(6(\gamma+\xi)))\Big). \end{aligned} \quad (37)$$

By combining the components, the 2nd order approximate HPM solution is presented as:

$$V_{HPM}(\gamma,\xi,\mu) = V_0(\gamma,\xi,\mu) + V_1(\gamma,\xi,\mu) + V_2(\gamma,\xi,\mu). \quad (38)$$

Solving Time Fractional FZK (3, 3, 3)

Consider the following equation

$$\frac{\partial^\beta V}{\partial \mu^\beta} + \frac{\partial V^3}{\partial \gamma} + 2\frac{\partial^3 V^3}{\partial \gamma^3} + 2\frac{\partial^3 V^3}{\partial \gamma \partial y^2} = 0, \qquad 0 < \beta \leq 1. \tag{39}$$

$$V(\gamma, \xi, 0) = \frac{3}{2}\lambda \sinh\Big(\frac{1}{6}(\gamma + \xi)\Big).$$

The exact solution of equation (39) for $\beta = 1$, is

$$V(\gamma, \xi, \mu) = \frac{3}{2}\lambda \sinh\Big(\frac{1}{6}(\gamma + \xi - \lambda\mu)\Big). \tag{40}$$

Application of NTIM

Applying natural transform on Equation (39), we get

$$N\Big(\frac{\partial^\beta V}{\partial \mu^\beta}\Big) = N\Big(-\frac{\partial V^3}{\partial \gamma} - 2\frac{\partial^3 V^3}{\partial \gamma^3} - 2\frac{\partial^3 V^3}{\partial \gamma \partial \xi^2}\Big). \tag{41}$$

By applying the differentiation property of the natural transform to Equation (41), we obtain the following result

$$\frac{s^\beta}{\nu^\beta}V(\gamma, \xi, \mu) - \frac{\nu^{\beta-1}}{s^\beta}V(\gamma, \xi, 0) = N\Big(-\frac{\partial V^3}{\partial \gamma} - 2\frac{\partial^3 V^3}{\partial \gamma^3} - 2\frac{\partial^3 V^3}{\partial \gamma \partial \xi^2}\Big). \tag{42}$$

By applying the inverse natural transform to Equation (42), we obtain

$$V(\gamma, \xi, \mu) = \frac{V(\gamma, \xi, 0)}{s} + N^-\Big\{\frac{\nu^\beta}{s^\beta}N\Big(-\frac{\partial V^3}{\partial \gamma} - 2\frac{\partial^3 V^3}{\partial \gamma^3} - 2\frac{\partial^3 V^3}{\partial \gamma \partial \xi^2}\Big)\Big\}. \tag{43}$$

Utilizing the recursive relation from Equation (15)

$$V_0(\gamma, \xi, \mu) = N^-\Big\{\frac{V(\gamma, \xi, 0)}{s}\Big\}.$$

$$V_1(\gamma, \xi, \mu) = N^-\Big\{\frac{\nu^\beta}{s^\beta}N\Big(-\frac{\partial V_0^3}{\partial \gamma} - 2\frac{\partial^3 V_0^3}{\partial \gamma^3} - 2\frac{\partial^3 V_0^3}{\partial \gamma \partial \xi^2}\Big)\Big\}.$$

$$\begin{aligned} V_2(\gamma, \xi, \mu) =& N^-\Big\{\frac{\nu^\beta}{s^\beta}N\Big\{\Big(-\frac{\partial (V_0+V_1)^3}{\partial \gamma} - 2\frac{\partial^3 (V_0+V_1)^3}{\partial \gamma^3} - 2\frac{\partial^3 (V_0+V_1)^3}{\partial \gamma \partial \xi^2}\Big) \\ & -\Big(-\frac{\partial V_0^3}{\partial \gamma} - 2\frac{\partial^3 V_0^3}{\partial \gamma^3} - 2\frac{\partial^3 V_0^3}{\partial \gamma \partial \xi^2}\Big)\Big\}. \end{aligned}$$

By using the software Wolfram Mathematica 13.2, the solution components are obtained as

$$V_0(\gamma, \xi, \mu) = \frac{3}{2}\lambda \sinh\left(\frac{\gamma+\xi}{6}\right). \tag{44}$$

$$V_1(\gamma, \xi, \mu) = \frac{3\lambda^3\mu^\beta\left(2\cosh\left(\frac{\gamma+\xi}{6}\right) + 3\left(-3\sinh\left(\frac{\gamma+\xi}{2}\right) + \sinh\left(\frac{\gamma+\xi}{6}\right) - 6\cosh\left(\frac{\gamma+\xi}{2}\right)\right)\right)}{128\Gamma(\beta+1)}. \tag{45}$$

$$V_2(\gamma,\xi,\mu) = \frac{3\lambda^5\mu^{2\beta}}{33554432}\Bigg(\frac{1}{\Gamma(2\beta+1)}\Big(4096(-2619\sinh(\frac{\gamma+\xi}{2}) + 94\sinh(\frac{\gamma+\xi}{6}) + 5175\sinh(\frac{5(\gamma+\xi)}{6})$$
$$- 2160\cosh(\frac{\gamma+\xi}{2}) + 96\cosh(\frac{\gamma+\xi}{6}) + 3600\cosh(\frac{5(\gamma+\xi)}{6}))$$
$$- \frac{64\lambda^2\Gamma(2\beta+1)\mu^\beta}{\Gamma(\beta+1)^2\Gamma(3\beta+1)}\Big(710\cosh(\frac{\gamma+\xi}{6}) + 3(5193\sinh(\frac{\gamma+\xi}{2}) + 425\sinh(\frac{\gamma+\xi}{6})$$
$$- 52275\sinh(\frac{5(\gamma+\xi)}{6}) + 93933\sinh(\frac{7(\gamma+\xi)}{6}) + 4230\cosh(\frac{\gamma+\xi}{2}) - 55950\cosh(\frac{5(\gamma+\xi)}{6})$$
$$+ 108486\cosh(\frac{7(\gamma+\xi)}{6}))) - \frac{\lambda^4\Gamma(3\beta+1)\mu^{2\beta}}{\Gamma(\beta+1)^3\Gamma(4\beta+1)} - 281412\sinh(\frac{\gamma+\xi}{2})$$
$$- 5727753\sinh(\frac{3(\gamma+\xi)}{2}) + 6298\sinh(\frac{\gamma+\xi}{6}) - 193500\sinh(\frac{5(\gamma+\xi)}{6}) + 1496313\sinh(\frac{7(\gamma+\xi)}{6})$$
$$- 224784\cosh(\frac{\gamma+\xi}{2}) - 5432508\cosh(\frac{3(\gamma+\xi)}{2}) + 5592\cosh(\frac{\gamma+\xi}{6}) - 262800\cosh(\frac{5(\gamma+\xi)}{6})$$
$$+ 1539972\cosh(\frac{7(\gamma+\xi)}{6}))\Big)\Bigg). \quad (46)$$

Combining these elements, the second-order approximate NTIM solution is formulated as follows:

$$V_{NTIM}(\gamma,\xi,\mu) = V_0(\gamma,\xi,\mu) + V_1(\gamma,\xi,\mu) + V_2(\gamma,\xi,\mu). \quad (47)$$

Application of HPM

Utilizing the HPM formulation outlined in Section 3, we obtain Zero-order component:

$$\frac{\partial^\beta V_0}{\partial\mu^\beta} = 0,$$
$$V(\gamma,\xi,0) = \frac{3}{2}\lambda\sinh\left(\frac{\gamma+\xi}{6}\right). \quad (48)$$

First-order component:

$$\frac{\partial^\beta V_1}{\partial\mu^\beta} = -\Bigg(6V_0^2\frac{\partial^3 V_0}{\partial\gamma\partial\xi^2} + 6V_0^2\frac{\partial^3 V_0}{\partial\gamma^3} + 12V_0\frac{\partial V_0}{\partial\gamma}\frac{\partial^2 V_0}{\partial\xi^2} + 24V_0\frac{\partial V_0}{\partial\xi}\frac{\partial^2 V_0}{\partial\gamma\partial\xi} + 36V_0\frac{\partial V_0}{\partial\gamma}\frac{\partial^2 V_0}{\partial\gamma^2}$$
$$+ 3V_0^2\frac{\partial V_0}{\partial\gamma} + 12\frac{\partial V_0}{\partial\gamma}\Big(\frac{\partial V_0}{\partial\xi}\Big)^2 + 12\Big(\frac{\partial V_0}{\partial\gamma}\Big)^3\Bigg).$$

Second-order component:

$$\frac{\partial^\beta V_2}{\partial\mu^\beta} = -\Bigg(6V_0^2\frac{\partial^3 V_1}{\partial\gamma\,\partial\xi^2} + 6V_0^2\frac{\partial^3 V_1}{\partial\gamma^3} + 12V_0V_1\frac{\partial^3 V_0}{\partial\gamma\,\partial\xi^2} + 24V_0\frac{\partial V_1}{\partial\xi}\frac{\partial^2 V_0}{\partial\gamma\,\partial\xi} + 12V_0\frac{\partial V_1}{\partial\gamma}\frac{\partial^2 V_0}{\partial\xi^2}$$
$$+ 24V_0\frac{\partial V_0}{\partial\xi}\frac{\partial^2 V_1}{\partial\gamma\,\partial\xi} + 12V_0\frac{\partial V_0}{\partial\gamma}\frac{\partial^2 V_1}{\partial\xi^2} + 24V_1\frac{\partial V_0}{\partial\xi}\frac{\partial^2 V_0}{\partial\gamma\,\partial\xi} + 12V_1\frac{\partial V_0}{\partial\gamma}\frac{\partial^2 V_0}{\partial\xi^2} + 12V_0V_1\frac{\partial^3 V_0}{\partial\gamma^3}$$
$$+ 36V_0\frac{\partial V_1}{\partial\gamma}\frac{\partial^2 V_0}{\partial\gamma^2} + 36V_0\frac{\partial V_0}{\partial\gamma}\frac{\partial^2 V_1}{\partial\gamma^2} + 36V_1\frac{\partial V_0}{\partial\gamma}\frac{\partial^2 V_0}{\partial\gamma^2} + 3V_0^2\frac{\partial V_1}{\partial\gamma} + 12\Big(\frac{\partial V_0}{\partial\xi}\Big)^2\frac{\partial V_1}{\partial\gamma}$$
$$+ 24\frac{\partial V_0}{\partial\gamma}\frac{\partial V_0}{\partial\xi}\frac{\partial V_1}{\partial\xi} + 6V_0V_1\frac{\partial V_0}{\partial\gamma} + 36\Big(\frac{\partial V_0}{\partial\gamma}\Big)^2\frac{\partial V_1}{\partial\gamma}\Bigg).$$

The solutions of above components are as follows:

$$V_0(\gamma,\xi,\mu) = \frac{3}{2}\lambda\sinh\left(\frac{\gamma+\xi}{6}\right). \quad (49)$$

$$V_1(\gamma,\xi,\mu) = -\frac{3\lambda^3\mu^\beta \cosh\left(\frac{\gamma+\xi}{6}\right)\left(9\cosh\left(\frac{\gamma+\xi}{3}\right)-7\right)}{16\beta\Gamma(\beta)}. \tag{50}$$

$$V_2(\gamma,\xi,\mu) = \frac{3\lambda^5\mu^{2\beta}\left(-621\sinh\left(\frac{\gamma+\xi}{2}\right)+70\sinh\left(\frac{\gamma+\xi}{6}\right)+765\sinh\left(\frac{5(\gamma+\xi)}{6}\right)\right)}{512\beta^2\Gamma(\beta)^2}. \tag{51}$$

Combining the components, the 2nd order approximate HPM solution is given as

$$V_{HPM}(\gamma,\xi,\mu) = V_0(\gamma,\xi,\mu) + V_1(\gamma,\xi,\mu) + V_2(\gamma,\xi,\mu). \tag{52}$$

RESULTS AND DISCUSSION

The FZK equation has been used to test the NTIM and HPM formulations; Mathematica 13.2 was used for the majority of the computational effort. Tables 1 and 2 present a comparison between the outcomes of the proposed method's second-order approximation for the FZK (2, 2, 2) equation and the third-order approximations derived from the VIM and the PIA, respectively. The FZK (2, 2, 2) equation's exact and approximate solutions, as determined by the suggested method, are plotted in three dimensions in Figures 3-4. Similarly, Figures 9 and 10 depict three-dimensional graphs comparing the exact and approximate solutions for the FZK (3, 3, 3) equation. In contrast, Figures 7 and 8 display two-dimensional illustrations of the FZK (3, 3, 3) equation at different β values, while Figures 1 and 2 show two-dimensional plots of the approximate solutions for the FZK (2, 2, 2) equation across varying β values. The two-dimensional graphics show that as β approaches 1, the approximate solutions become more similar to the exact solutions. The suggested method accurately approximates solutions for various configurations of the FZK problem, as evidenced by its convergence. Visual comparisons demonstrate the method's ability to handle higher-order equations and variable parameters, indicating its promise for solving complicated differential equations.

γ	ξ	μ	V_{NTIM}	V_{HAM}	V_{Exact}	$NTIM\ Error$	$HPM\ Error$	$PIA\ Error$ [36]
0.1	0.1	0.2	5.20963×10^{-5}	5.20977×10^{-5}	5.25222×10^{-5}	4.25938×10^{-7}	4.24472×10^{-7}	3.85217×10^{-7}
0.1	0.1	0.3	5.18325×10^{-5}	5.18358×10^{-5}	5.24703×10^{-5}	6.37842×10^{-7}	6.34545×10^{-7}	5.75911×10^{-7}
0.1	0.1	0.4	5.15695×10^{-5}	5.15753×10^{-5}	5.24185×10^{-5}	8.49036×10^{-7}	8.43176×10^{-7}	7.65359×10^{-7}
0.6	0.6	0.2	1.15981×10^{-3}	1.15982×10^{-3}	1.15808×10^{-3}	1.72413×10^{-6}	1.73295×10^{-6}	4.66337×10^{-5}
0.6	0.6	0.3	1.16059×10^{-3}	1.1606×10^{-3}	1.15799×10^{-3}	2.59286×10^{-6}	2.61268×10^{-6}	6.86056×10^{-5}
0.6	0.6	0.4	1.16137×10^{-3}	1.1614×10^{-3}	1.1579×10^{-3}	3.46604×10^{-6}	3.50126×10^{-6}	8.98263×10^{-5}
0.9	0.9	0.2	1.26485×10^{-3}	1.26484×10^{-3}	1.26462×10^{-3}	2.23227×10^{-7}	2.15846×10^{-7}	5.12131×10^{-4}
0.9	0.9	0.3	1.26501×10^{-3}	1.265×10^{-3}	1.26468×10^{-3}	3.29329×10^{-7}	3.12727×10^{-7}	7.38186×10^{-4}
0.9	0.9	0.4	1.26517×10^{-3}	1.26514×10^{-3}	1.26474×10^{-3}	4.31752×10^{-7}	4.02247×10^{-7}	9.57942×10^{-4}

Table 1: For FZK (2, 2, 2) at λ = 0.001 and β = 1, the second-order approximate solution obtained by NTIM and HPM is compared to the PIA solution.

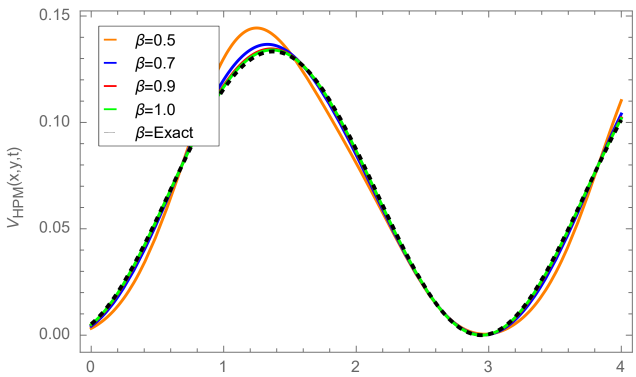

Figure 1: Solution profile of $V(\gamma, \xi, \mu)$ with differ-ent β values when $\xi = 0.2$, $\lambda = 0.1$ and $\mu = 0.01$ for problem 1 using HPM

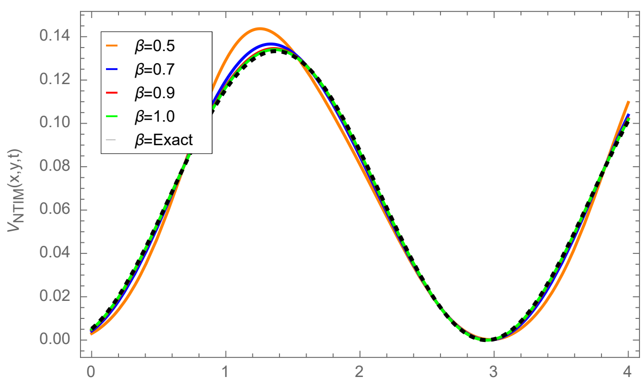

Figure 2: Solution profile of $V(\gamma, \xi, \mu)$ with differ-ent β values when $\xi = 0.2$, $\lambda = 0.1$ and $\mu = 0.01$ for problem 1 using NTIM

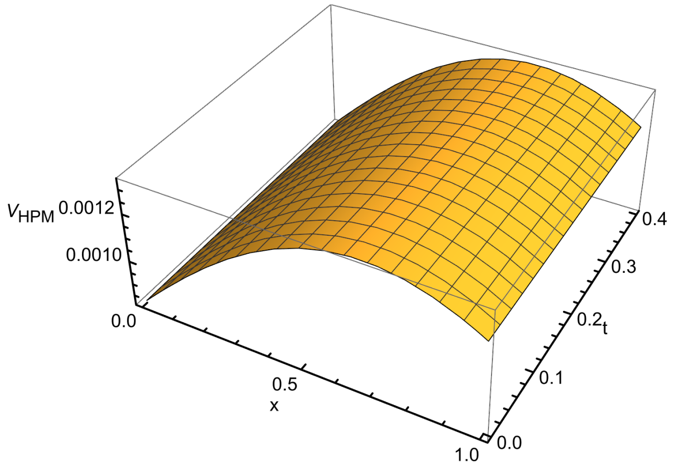

Figure 3: 3D profile of V (γ, ξ, μ) with β = 1, ξ = 0.9, λ = 0.001 for problem 1 using HPM

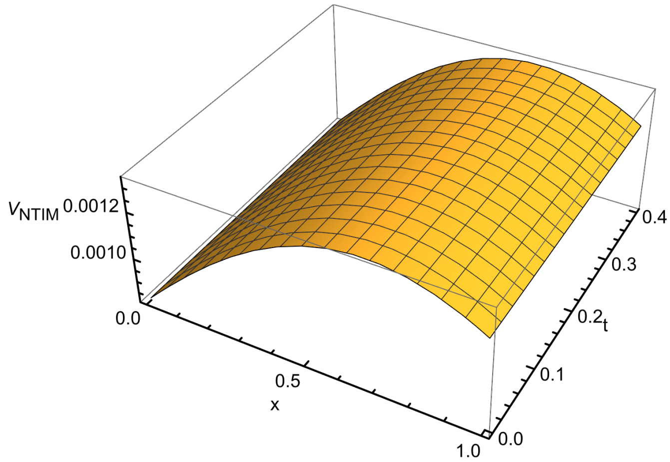

Figure 4: 3D profile of V (γ, ξ, μ) with β = 1, ξ = 0.9, λ = 0.001 for problem 1 using NTIM

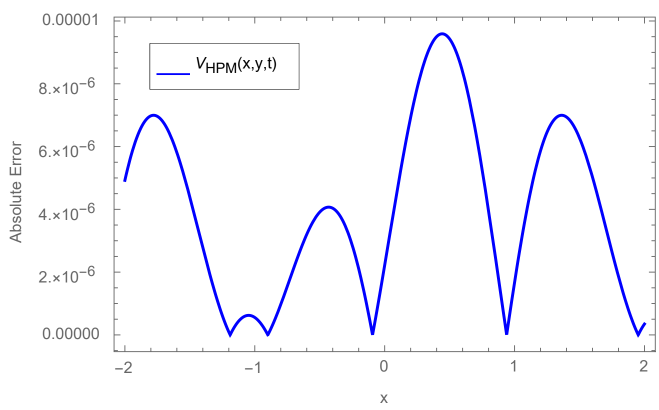

Figure 5: Absolute error graph of V (γ, ξ, μ), when ξ = 0.9, λ = 0.01, β = 1 and μ = 0.01 for problem 1 using HPM

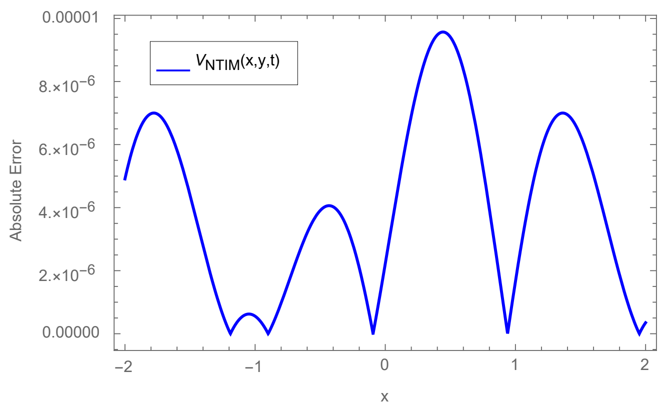

Figure 6: Absolute error graph of V (γ, ξ, μ), when ξ = 0.9, λ = 0.01, β = 1 and μ = 0.01 for problem 1 using NTIM

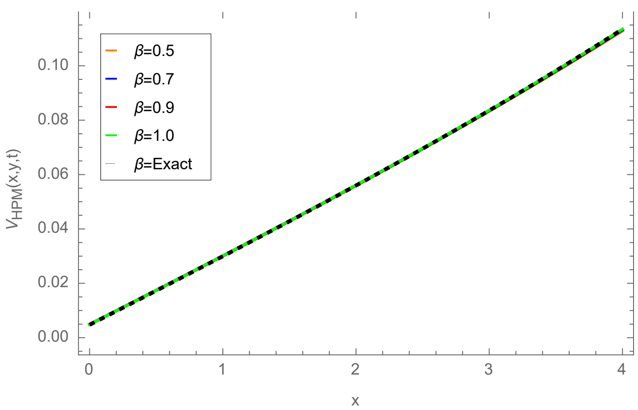

Figure 7: Solution profile of V (γ, ξ, μ) with differ-ent β values when ξ = 0.2, λ = 0.1 and μ = 0.1 for problem 2 using HPM

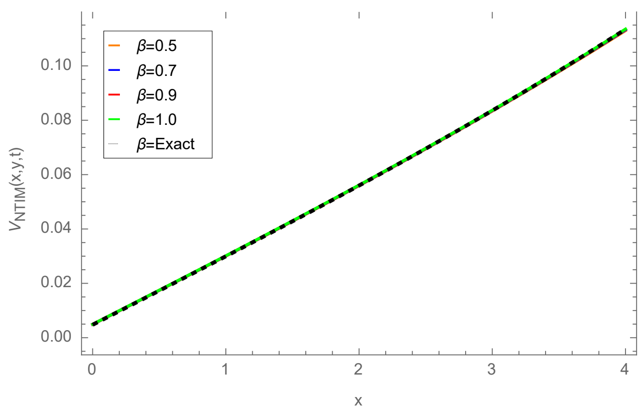

Figure 8: Solution profile of V (γ, ξ, μ) with differ-ent β values when ξ = 0.2, λ = 0.1 and μ = 0.1 for problem 2 using NTIM

γ	ξ	μ	V_{NTIM}	V_{HPM}	V_{VIM} [36]	V_{Exact}	$NTIM\ Error$	$HPM\ Error$
0.1	0.1	0.2	5.00092×10^{-5}	5.00092×10^{-5}	5.00091×10^{-5}	4.99592×10^{-5}	4.99486×10^{-8}	4.9952×10^{-8}
0.1	0.1	0.3	5.00091×10^{-5}	5.00091×10^{-5}	5.00091×10^{-5}	4.99342×10^{-5}	7.49228×10^{-8}	7.49279×10^{-8}
0.1	0.1	0.4	5.00091×10^{-5}	5.00091×10^{-5}	5.00091×10^{-5}	4.99092×10^{-5}	9.98971×10^{-8}	9.99039×10^{-8}
0.6	0.6	0.2	3.02004×10^{-4}	3.02004×10^{-4}	3.02003×10^{-4}	3.01953×10^{-4}	5.08887×10^{-8}	5.08988×10^{-8}
0.6	0.6	0.3	3.02004×10^{-4}	3.02004×10^{-4}	3.02003×10^{-4}	3.01927×10^{-4}	7.63329×10^{-8}	7.6348×10^{-8}
0.6	0.6	0.4	3.02004×10^{-4}	3.02004×10^{-4}	3.02003×10^{-4}	3.01902×10^{-4}	1.01777×10^{-7}	1.01797×10^{-7}
0.9	0.9	0.2	4.5678×10^{-4}	4.5678×10^{-4}	4.56780×10^{-4}	4.56728×10^{-4}	5.21165×10^{-8}	5.21228×10^{-8}
0.9	0.9	0.3	4.5678×10^{-4}	4.5678×10^{-4}	4.56780×10^{-4}	4.56702×10^{-4}	7.81746×10^{-8}	7.81841×10^{-8}
0.9	0.9	0.4	4.5678×10^{-4}	4.5678×10^{-4}	4.56780×10^{-4}	4.56676×10^{-4}	1.04233×10^{-7}	1.04245×10^{-7}

Table 2: At $\lambda = 0.001$ and $\beta = 1$, the second-order approximation from NTIM and HPM is compared with the VIM solution for FZK (3, 3, 3).

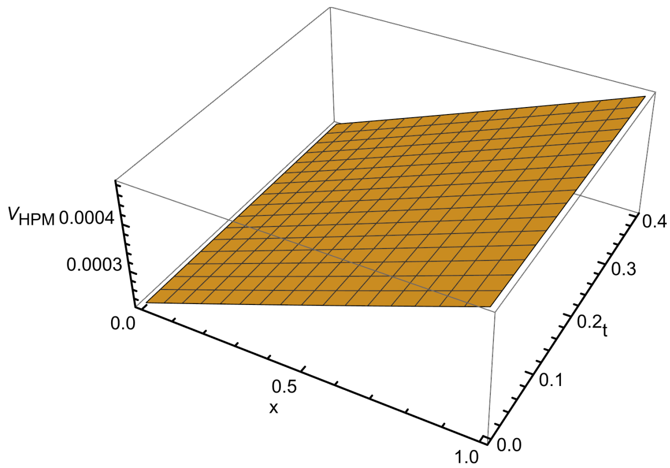

Figure 9: 3D profile of $V(\gamma, \xi, \mu)$ with $\beta = 1$, $\xi = 0.9$, $\lambda = 0.001$ for problem 2 using HPM

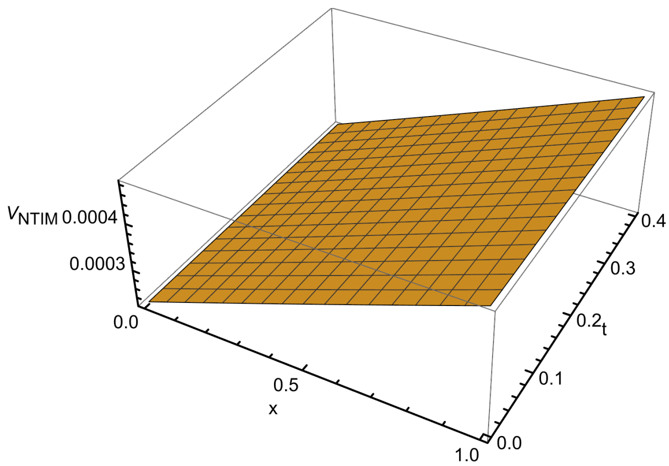

Figure 10: 3D profile of $V(\gamma, \xi, \mu)$ with $\beta = 1$, $\xi = 0.9$, $\lambda = 0.001$ for problem 2 using NTIM

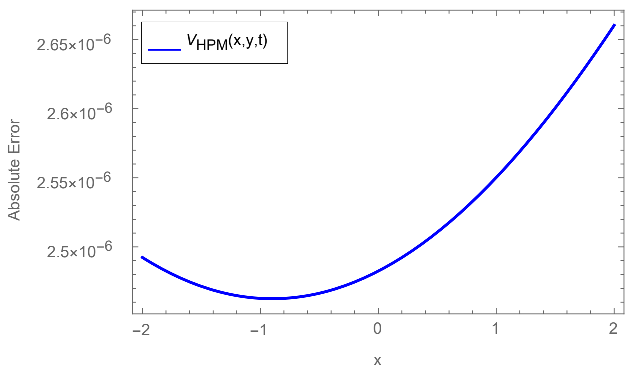

Figure 11: Absolute error graph of $V(\gamma, \xi, \mu)$, when $\xi = 0.9$, $\lambda = 0.01$, $\beta = 1$ and $\mu = 0.1$ for problem 2 using HPM

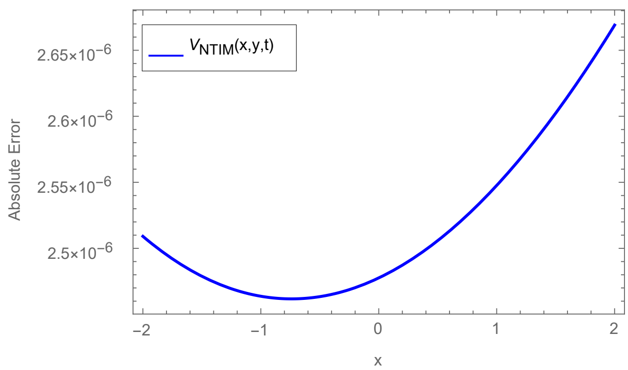

Figure 12: Absolute error graph of $V(\gamma, \xi, \mu)$, when $\xi = 0.9$, $\lambda = 0.01$, $\beta = 1$ and $\mu = 0.1$ for problem 2 using NTIM

CONCLUSION

Second-order NTIM and HPM have produced more promising results in solving FPDE than third-order approximations such as PIA and VIM. The obtained findings demonstrate the efficacy and ease of the proposed methodology for tackling higher-dimensional FPDEs. The data show that the second-order NTIM and HPM solutions perform better, indicating that they are capable of dealing with complicated mathematical problems. The findings highlight the method's effectiveness and ease in tackling difficulties associated with higher-dimensional FPDEs. The proposed method's accuracy can be improved by using higher-order approximations, allowing for more precise solution techniques. This technique is consistent with the ongoing search of advanced numerical methodologies for address-ing complex mathematical problems encountered in a variety of scientific and engineering disciplines.

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