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The Boussinesq Equation: Bidirectional Shallow-Water Waves, Solitary Waves, and the Paradigm of Nonlinear Dispersive Modelling – A Comprehensive Review

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ABSTRACT

The Boussinesq equation, introduced by Joseph Boussinesq in 1872, was the first mathematical model to explain the existence of the solitary wave observed by John Scott Russell in 1834, and it remains the canonical model for bidirectional, weakly nonlinear, weakly dispersive wave propagation in shallow water and other media. Unlike the unidirectional Kortewegde Vries equation, the Boussinesq equation allows waves to travel in both directions and captures the leading-order balance between nonlinearity and dispersion in a single second-order-in-time evolution equation. Over the past century and a half it has grown into an enormous family of models: the classical, improved, regularized, and generalized Boussinesq equations; the Boussinesq systems for surface and internal waves; the Boussinesq paradigm equation in two dimensions; and the Boussinesq-Burgers, logarithmic Boussinesq, fractional, and stochastic variants. This review provides a comprehensive and critical assessment of the literature. We survey the physical derivations from the water-wave problem, the theory of solitary-wave existence, stability, and blow-up, the well-posedness theory of the good and bad Boussinesq equations, numerical methods including conservative finite-difference, finite-element, and spectral schemes, the Boussinesq-type models used in coastal engineering and tsunami modelling, and the many exact-solution families obtained by Hirota, Wazwaz, and others. Recent contributions from 2025 and 2026 are highlighted throughout, and the review concludes with a discussion of open problems and future directions.

Keywords: Boussinesq Equation, Solitary Waves, Shallow-Water Waves, Nonlinear, Dispersive

INTRODUCTION

The Boussinesq equation occupies a special place in the history of nonlinear science. In 1834, John Scott Russell observed a solitary wave of translation travelling along a canal near Edinburgh, and his report launched a century of debate about the existence and nature of such waves [1]. In 1872, Boussinesq derived the first mathematical model capable of describing the solitary wave as a persistent, localized solution, thereby founding the theory of nonlinear dispersive wave equations [1-2]. His equation,

$$u_{tt} = u_{xx} + (u^2)_{xx} + u_{xxxx},$$

combines a nonlinear term, a dispersive term, and a second time derivative, so that waves propagate in both directions. It was the first equation to balance nonlinearity and dispersion in a way that supports stable solitary waves, and it predates the Korteweg-de Vries equation of 1895, which describes the unidirectional counterpart of the same physical system [1,3].

The Boussinesq equation is best understood as the bidirectional companion of the KdV equation. Whereas the KdV equation describes waves travelling in one direction and arises from the water-wave problem by factoring the linear wave operator, the Boussinesq equation retains both factors and hence allows left- and right-moving waves. Its derivation from the full water-wave problem, and the rigorous justification of the reduction, has been studied by many authors, beginning with Boussinesq himself and continuing through the modern theory of asymptotic models for water waves [2,4-5]. The equation is also the long-wave limit of the Fermi-Pasta-Ulam-Tsingou lattice, connecting the continuum theory of shallow-water waves with the discrete dynamics of nonlinear lattices [6-8]. This dual origin, in both continuum fluid mechanics and discrete lattice dynamics, is responsible for the extraordinary breadth of its applications.

The mathematical structure of the Boussinesq family is subtle. The classical equation is linearly ill-posed, and for this reason it is known as the bad Boussinesq equation; on the contrary, the good Boussinesq equation,

$$u_{tt} = u_{xx} + (u^2)_{xx} - u_{xxxx},$$

with the opposite sign of the dispersive term, is linearly well-posed and is the object of most of the rigorous well-posedness and stability theory [9-10]. The distinction between good and bad Boussinesq equations is one of the recurring themes of the subject, and a large literature has grown up around the regularization and improvement of the equations to achieve well-posedness while preserving the physically correct dispersion [11-13]. The purpose of this review is to survey this landscape critically and comprehensively.

The structure of the review is as follows. Section 2 describes the methodology used to identify and select the literature. Section 3 recalls the physical and historical origins of the equation. Section 4 treats the solitary-wave theory: existence, orbital stability, and the stability-instability dichotomy. Section 5 reviews well-posedness and ill-posedness. Section 6 covers blow-up and global existence. Section 7 surveys the Boussinesq system family for surface and internal waves. Section 8 reviews numerical methods. Section 9 examines the Boussinesq paradigm equation in two dimensions. Section 10 covers the Boussinesq-type models used in coastal engineering and tsunami science. Section 11 surveys exact and explicit solutions, including the logarithmic variants. Section 12 treats the fractional and coupled variants. Section 13 describes the lattice and discrete origins. Section 14 collects open problems and future directions, and Section 15 concludes.

REVIEW METHODOLOGY

This review was assembled through a systematic search of the literature on the Boussinesq equation and its relatives. The search was carried out in the bibliographic databases Scopus, Web of Science, MathSciNet, zbMATH, and Google Scholar, with the arXiv preprint server (sections nlin.PS and math.AP) consulted for the most recent and forthcoming work. The reference lists of the retrieved articles, and of the standard monographs and review articles on nonlinear dispersive waves, were examined in turn, so that a backward snowballing step could recover earlier or more specialised contributions that the keyword search alone did not return.

The search strings combined the term “Boussinesq” with one or more of the following terms: solitary wave, shallow water, nonlinear dispersive, well-posedness, ill-posedness, blow-up, stability, Boussinesq system, internal wave, Boussinesq paradigm, tsunami, coastal engineering, exact solution, Hirota, fractional, logarithmic, Boussinesq-Burgers, and Fermi-Pasta-Ulam. Boolean operators were used to combine the terms, and truncation was applied where the database syntax permitted it.

No lower limit was placed on the publication date, so that the historical origins of the subject could be represented; the earliest work retained is from 1976 and the most recent from 2026. Particular attention was paid to the literature of the last decade, and works accepted or in press at the time of writing were included when a digital object identifier was available. A study was included if it introduced, derived, analysed, or applied the Boussinesq equation or one of its variants and generalisations, or if it reported a numerical method, an exact solution, or a physical application directly relevant to that family, and peer-reviewed journal articles, monographs, and conference proceedings were prioritised. A study was excluded if the Boussinesq equation appeared only as a passing comparison, if the full text could not be obtained, or if the work was a duplicate or a preliminary version superseded by a later publication.

The retained works were then grouped by theme — derivation and history, solitary-wave theory, well-posedness, blow-up and global existence, systems and internal waves, numerical methods, the two-dimensional paradigm equation, coastal and tsunami models, exact solutions, fractional and coupled variants, and lattice origins — and each section of the review corresponds to one of these themes. No quantitative meta-analysis was attempted, because the literature is heterogeneous in method and scope; the review is accordingly a critical synthesis rather than a systematic quantitative survey.

Physical and Historical Origins

The story of the Boussinesq equation begins with the solitary wave observed by Russell in 1834 and reported in 1844 [1]. Russell's claim that such waves could propagate without change of form was contested, notably by Stokes, and the resolution required a mathematical theory of nonlinear dispersive waves. Boussinesq supplied this theory in 1872, deriving an equation for the surface elevation that supports solitary-wave solutions of permanent form [1-2]. His derivation was based on the assumption of a small-amplitude, long-wavelength approximation to the Euler equations, an approximation scheme that remains the foundation of modern asymptotic water-wave theory [14-15]. The solitary-wave phenomenon and the classical nonlinear evolution equations are surveyed in the historical introduction of David and in the monographs of Drazin and Johnson and of Ablowitz and Clarkson [16-18]. Kano and Nishida later provided a rigorous justification of both the KdV and Boussinesq reductions from the full Euler equations [2].

The derivation of the Boussinesq equation illustrates the general structure of asymptotic models for water waves. Under the scaling in which the amplitude is small and the wavelength is long, the free-surface problem can be reduced to a hierarchy of shallow-water models: the nonlinear shallow-water equations, the Boussinesq equations, and the KdV equation, depending on the relative order of the nonlinearity and dispersion that is retained [14-15]. The Hamiltonian structure of the full water-wave problem, developed by Zakharov and others, provides a natural framework for deriving these models in a consistent way, and Craig and Groves constructed Hamiltonian long-wave approximations to the water-wave problem that include the Boussinesq systems [4]. The modern theory of

asymptotic models for surface and internal waves, developed by Bona, Lannes, and Saut, provides rigorous error estimates for the Boussinesq and related approximations over physically relevant time scales [19].

The Boussinesq equation is also the natural continuum limit of the Fermi-Pasta-Ulam-Tsingou lattice, the one-dimensional chain of nonlinearly coupled masses that played a central role in the history of nonlinear science [8]. The equations of motion of the lattice reduce, in the long-wave limit, to a Boussinesq-type equation for the counter-propagating waves, and the rigorous analysis of this limit was carried out by Schneider and Wayne, who proved that solutions of the lattice dynamics are well approximated by solutions of the continuum equations over long time scales [7]. The travelling-wave theory of the FPU lattice, developed by Iooss and others, provides a deep connection between the discrete and continuum descriptions [6]. This lattice connection is not merely historical: it continues to motivate research on solitary waves, breathers, and energy transport in nonlinear chains, and it demonstrates that the Boussinesq equation is a universal model for bidirectional nonlinear wave propagation in dispersive media [20-21].

The historical and conceptual importance of the Boussinesq equation is thus twofold: it is the first successful mathematical model of the solitary wave, and it is the prototype of the systematic derivation of nonlinear dispersive models from first principles [1-2]. The modern literature, from the Hamiltonian formulations of the water-wave problem to the rigorous justification of asymptotic models, stands directly on this foundation [4-5,19].

Solitary Waves: Existence and Stability

The solitary-wave solutions of the Boussinesq equation are the most important objects in its theory. For the generalized Boussinesq equation

$$u_{tt} = u_{xx} + (f(u))_{xx} - u_{xxxx},$$

solitary waves are obtained from the travelling-wave reduction, which leads to an ODE for the wave profile that can be solved explicitly for power-law nonlinearities [9,22]. The existence of solitary waves for a wide class of nonlinearities, and their characterization as minimizers of an energy functional, was established by Bona and Sachs in their study of the generalized Boussinesq equation [9]. Linares proved the global existence of small solutions for the generalized equation, complementing the solitary-wave theory [23]. The ground states are radially symmetric, exponentially decaying, and smooth, and their speed-amplitude relation depends on the nonlinearity [9,24]. The solitary-wave interactions of the Boussinesq equation were examined numerically by Bona, Pritchard, and Scott, who showed that the waves do not emerge from the interaction unscathed but generate two new solitary waves together with a small dispersive tail [25], and their evaluation of the model equation for water waves against laboratory experiments established the quantitative validity of the Boussinesq description [26].

The stability of Boussinesq solitary waves is governed by the interplay between the conserved quantities and the spectral properties of the linearized operator. Bona and Sachs proved the orbital stability of solitary waves for the generalized Boussinesq equation under a convexity condition on the energy [9], and the abstract stability theory of Grillakis, Shatah, and Strauss provides the general framework for this and related results [24,27]. Liu proved the instability of solitary waves for generalized Boussinesq equations in regimes where the convexity condition fails [22], and obtained sharp conditions for the finite-time blow-up of solutions in terms of the energy of the ground state [28]. Pego and Weinstein developed the theory of convective linear stability for the good Boussinesq equation, showing that the linearized evolution around a solitary wave decays in appropriate weighted norms and establishing the nonlinear asymptotic stability of the waves [29]. The stability theory for the Boussinesq-KdV systems, in which the Boussinesq equation is coupled to a KdV-type equation, was developed by Quintero and others [30].

The distinction between the stability and instability regimes is sharp and physically significant. For the power-law nonlinearity u^p, the critical power that separates stability from instability is determined by the scaling of the conserved quantities, in direct analogy with the nonlinear Schrödinger and generalized KdV equations [9,22,24]. The stability analysis of the Boussinesq paradigm equation, and its two-dimensional solitary waves, was carried out by Christou and Christov using spectral methods [31], and the theoretical and numerical aspects of global existence and blow-up for the paradigm equation were studied by Kutev, Kolkovska, Dimova, and Christov [32]. The recent results of Dimova, Kolkovska, and Kutev on orbital stability for the generalized Boussinesq equation with quadratic-cubic nonlinearity complete the picture for the physically relevant combined nonlinearities [33]. These results establish the Boussinesq solitary waves as a canonical example of the stability theory of nonlinear dispersive waves [27,34].

Well-Posedness and Ill-Posedness

The well-posedness theory of the Boussinesq equations is delicate because of the two signs of the dispersive term. The classical (bad) Boussinesq equation, with +u_xxxx, has a dispersion relation that makes the linearized problem ill-posed in any reasonable space, and this is the origin of its name [11,13]. The good Boussinesq equation, with -u_xxxx, is linearly well-posed, and its nonlinear well-posedness has been established in a series of papers. Bona and Sachs proved the global existence of smooth solutions and the stability of solitary waves for the generalized Boussinesq equation [9], and Linares proved the global existence of small solutions [23]. Liu established the blow-up and instability theory [22,28]. The modern well-posedness theory in Sobolev spaces was developed by Kishimoto and Tsugawa, who proved local well-posedness for the good Boussinesq equation in spaces below the energy space [35], and Kishimoto later proved sharp local well-posedness for the good Boussinesq equation in H^s for s > -1/2 [10].

The nonlocal models introduced by Abdelouhab, Bona, Felland, and Saut provide a rigorous framework for the ill-posed classical Boussinesq equation and for the family of regularized and improved equations that are well-posed [11]. Bona and Chen studied the well-posedness of regularized nonlinear dispersive wave equations, establishing global well-posedness for the improved Boussinesq family [36], and their comparison of model equations for small-amplitude long waves clarified the relationships among the KdV, BBM, and Boussinesq models [37]. Wang and Esfahani proved well-posedness for the Cauchy problem associated to a periodic Boussinesq equation [38]. The Boussinesq systems for bidirectional propagation, introduced by Bona and Chen, were shown to be well-posed in a range of Sobolev spaces, and the derivation and linear theory of the full family of Boussinesq systems was developed by Bona, Chen, and Saut [13,39]. Kita and Segata proved well-posedness for a Boussinesq-type system related to the water-wave problem [40]. Bellec and Colin studied the existence of solitary waves for Boussinesq-type equations and the Cauchy problem for a new conservative model [41]. The sharp results of Kishimoto, and the systematic theory of the Boussinesq systems, make the well-posedness of the good variants essentially complete [10,39].

The ill-posedness of the bad Boussinesq equation has a constructive counterpart: the equation serves as a useful model despite its linear ill-posedness, and the exact-solution literature provides a large family of explicit solutions that are meaningful as formal solutions of the equation [42-44]. The ill-posed Boussinesq equation arises in shallow-water waves and nonlinear lattices, and its Lie symmetries, conservation laws, and exact solutions have been studied extensively [42]. The dynamics of solitons to the ill-posed Boussinesq equation, including the family of bell-shaped, singular, and topological solitons, was examined by Tchier, Aliyu, Yusuf, and Inc [43], and the auxiliary equation method was applied to the ill-posed equation by Bibi and collaborators [44]. These studies show that the ill-posed equation, while not the object of the rigorous well-posedness theory, remains a valuable source of exact solutions and physical insight [42,44].

Blow-Up and Global Existence

The global existence and blow-up theory of the Boussinesq equations is governed by the conserved energy. For the generalized Boussinesq equation

$$u_{tt} = u_{xx} + (u^{p+1}/(p+1))_{xx} - u_{xxxx},$$

the energy and the momentum are conserved, and the dynamics are determined by the balance between the kinetic and potential parts of the energy [9,28]. Bona and Sachs proved global existence of smooth solutions for the subcritical powers and for data near the ground states [9]. Linares proved global existence of small solutions for the generalized equation [23]. Liu proved that solutions with energy below the ground-state threshold remain bounded, while solutions with energy above the threshold can blow up in finite time; in particular, there exist initial data arbitrarily close to the stationary state of lowest energy whose solutions blow up [28].

The blow-up theory of the Boussinesq paradigm equation, which arises in two-dimensional applications, was developed by Kutev, Kolkovska, Dimova, and Christov, who combined the variational analysis with careful numerical computation of the blow-up solutions [32]. Kolkovska and Todorov developed conservative finite-difference schemes for the multidimensional Boussinesq equation, which preserve the invariants of the continuous equation and are essential for the accurate simulation of both global existence and blow-up dynamics [45-46]. The well-posedness and blow-up theory for the integrable nonlinearly dispersive model equation studied by Li and Olver provides a related example in which the equation admits both global solutions and finite-time blow-up [47]. The damped Boussinesq equation, which adds a dissipation term to the model, was studied by Varlamov, who constructed classical solutions and described the large-time asymptotics in terms of two counter-propagating solitons governed by Burgers equations [48].

The sharp results on global existence and blow-up have been extended in several directions. Taskesen and Polat proved the existence of global solutions for a multidimensional Boussinesq-type equation with supercritical initial energy [49], and Dai and Chen proved global well-posedness for the Cauchy problem of generalized Boussinesq equations in the control problem regarding initial data [50]. Zhang, Zhang, and Niu established the global well-posedness of the Cauchy problem for damped multidimensional generalized Boussinesq equations with special nonlinear terms [51]. These results, together with the classical theory, provide a fairly complete picture of the global dynamics of the Boussinesq family [9,28,32].

Boussinesq Systems for Surface and Internal Waves

The single-equation Boussinesq model has a natural generalization to systems of two equations, one for the surface elevation and one for the horizontal velocity, which are equivalent to the single equation at the relevant order but possess better mathematical properties [12-13]. Bona and Chen introduced a Boussinesq system for two-way propagation of nonlinear dispersive waves and proved its well-posedness [12]. Bona, Chen, and Saut developed the systematic theory of Boussinesq systems, deriving the family of systems from the water-wave problem and studying their linear and nonlinear well-posedness [13,39]. The KdV-KdV systems of Boussinesq type, which are important in internal-wave applications, were studied numerically by Bona, Dougalis, and Mitsotakis [52], and their asymptotic behavior was analyzed by Capistrano-Filho and Gallego [53]. Bona, Chen, and Saut also proved well-posedness for regularized nonlinear dispersive wave equations, completing the theory for the improved systems [36].

For internal waves in stratified fluids, Boussinesq systems arise naturally from the two-layer and continuously stratified formulations. Albert, Bona, and Saut derived model equations for waves in stratified fluids and studied their well-posedness [54]. Bona, Ponce,

Saut, and Tom introduced a model system for strong interaction between internal solitary waves, and analyzed its solitary-wave solutions and their stability [55]. Anh studied the Boussinesq/full-dispersion systems and the Boussinesq/Boussinesq systems for internal waves [56]. Bona, Lannes, and Saut developed the rigorous theory of asymptotic models for internal waves, providing error estimates for the Boussinesq and related systems over the relevant time scales [19]. The internal-wave applications of the Boussinesq theory, from the generation of internal waves by topography to their propagation and disintegration, connect the mathematical models to oceanographic observation [57].

The Boussinesq systems are also the natural setting for the rigorous derivation of the KdV approximation. Duchêne proved that the Boussinesq/Boussinesq systems for internal waves with a free surface reduce to the KdV equation in the appropriate regime, with rigorous error estimates [58]. This result is the internal-wave analogue of the classical justification of the KdV reduction from the Boussinesq equation for surface waves, and it demonstrates the hierarchical structure of the water-wave approximations [4-5]. The Hamiltonian structure of the water-wave problem, and its long-wave approximations, provide the unifying framework within which the Boussinesq systems are derived and analyzed [4,14].

Numerical Methods

The numerical solution of the Boussinesq equations has a long history, motivated both by the need to validate the theory and by the demands of coastal engineering. Because the equation preserves several invariants and supports both stable solitary waves and blow-up solutions, the design of conservative and stable numerical schemes is essential [32,45]. Early finite-difference schemes for the good and bad Boussinesq equations were developed by El-Zoheiry, who studied the solitary-wave interactions numerically [59-60]. Bratsos constructed a second-order numerical scheme for the one-dimensional Boussinesq equation, providing accurate solutions for the solitary-wave propagation [61]. The Galerkin finite-element method with cubic B-splines was applied to the good and bad Boussinesq equations by Ucar, Esen, and Karaagac, who computed the solitary-wave motion, the interaction of solitary waves, and the blow-up solutions [62].

The development of conservative schemes for the multidimensional Boussinesq paradigm equation has been a major theme. Kolkovska constructed four-level conservative finite-difference schemes that preserve the discrete analogues of the invariants of the continuous equation [63], and proved error estimates for these schemes in the multidimensional setting [45]. Kolkovska, Todorov, and Christov developed two families of finite-difference schemes for the multidimensional Boussinesq paradigm equation, analyzing their stability and convergence [46]. Christou and Christov developed a Galerkin spectral method for the two-dimensional solitary waves of the Boussinesq paradigm equation [31]. These conservative and spectral methods are essential for the accurate long-time simulation of the paradigm equation, including the study of blow-up and of solitary-wave interactions [31-32].

Meshless and spectral methods provide complementary approaches. Dehghan and Salehi developed a meshless technique based on radial basis functions for the traveling solitary-wave solution of the Boussinesq equation, achieving high accuracy with collocation methods [64]. The numerical solution of the KdV-KdV systems of Boussinesq type was studied by Bona, Dougalis, and Mitsotakis, who combined the spectral and finite-element approaches for the internal-wave systems [52]. The fractional and higher-order variants of the Boussinesq equation have also been treated numerically; Ari and Dereli developed efficient numerical approaches, based on radial basis function collocation and quintic B-spline methods, for the time-fractional good Boussinesq equation [65]. These numerical developments support both the fundamental analysis and the engineering applications of the Boussinesq family [62,64-65].

The Boussinesq Paradigm Equation

The Boussinesq paradigm equation, proposed by Christov and collaborators, is a regularized two-dimensional generalization of the classical Boussinesq equation that supports stable solitary waves and exhibits a rich phenomenology of two-dimensional interactions [32,66]. The equation combines the bidirectional propagation of the classical model with a regularization of the dispersion that makes it well-posed and amenable to long-time simulation. The theoretical and numerical analysis of the paradigm equation, including the study of global existence, blow-up, and solitary-wave interactions, was carried out by Kutev, Kolkovska, Dimova, and Christov [32]. Todorov's monograph develops the two-dimensional Boussinesq equation as a paradigm of nonlinear wave modelling, presenting the soliton solutions and their interactions in detail [66].

The numerical study of the paradigm equation has driven the development of conservative schemes and of high-accuracy spectral methods. Christou and Christov applied the Galerkin spectral method to compute the two-dimensional solitary waves and their interactions [31]. Kolkovska and collaborators developed the family of four-level conservative finite-difference schemes and proved their convergence in the multidimensional setting [45,63]. Kolkovska, Todorov, and Christov compared two families of finite-difference schemes for the multidimensional paradigm equation, establishing the conservation properties and the stability of the schemes [46]. These numerical tools have made it possible to study the collision of two-dimensional solitary waves, the formation of bound states, and the blow-up of solutions in the paradigm model [32,66].

The two-dimensional solitary waves of the Boussinesq paradigm equation exhibit properties that differ qualitatively from their one-dimensional counterparts. Unlike the one-dimensional case, where the solitary waves are the unique ground states, the two-dimensional paradigm equation supports a family of localized solutions whose shape depends on the amplitude and on the direction of propagation, and whose interactions can be inelastic [31,66]. The Boussinesq paradigm equation has therefore become a standard

testbed for numerical methods for multidimensional nonlinear dispersive wave equations, and the exact and approximate solutions constructed for it serve as benchmarks for the conservative and spectral schemes reviewed in Section 8 [31,45].

Boussinesq-Type Models in Coastal Engineering and Tsunami Science

Boussinesq-type models are among the most widely used tools in coastal engineering for the simulation of nearshore wave dynamics. The models retain the leading-order balance of nonlinearity and dispersion, but are enhanced to improve their linear dispersion properties so that they can be applied to intermediate water depths and to the surf zone [67-68]. Madsen and Schäffer developed a systematic enhancement of Boussinesq-type equations, improving the dispersion relation and the shoaling properties of the models [68], and their comprehensive review of Boussinesq-type equations for surface gravity waves remains the standard reference for the subject [67]. Sørensen, Schäffer, and Madsen developed a Boussinesq-type model for the surf zone, simulating wave-induced horizontal nearshore circulations [69]. Lynett developed high-order Boussinesq-type equations for nearshore wave modeling, extending the range of validity of the models [70].

Tsunami modelling has become one of the most important applications of Boussinesq-type models. Fuhrman and Madsen developed a high-order Boussinesq model for tsunami generation, propagation, and run-up, demonstrating the capability of the models to reproduce the full tsunami lifecycle from the source to the coast [71]. The run-up of solitary waves and of realistic tsunamis on beaches and coastal structures, computed with Boussinesq-type models, has been compared with laboratory experiments and with nonlinear shallow-water models, establishing the accuracy and the limitations of the Boussinesq approach [67,71]. The hybrid Boussinesq-NSWE solvers, which couple the Boussinesq equations offshore with the nonlinear shallow-water equations near the coast, extend the range of the models to the swash zone [70]. These engineering applications demonstrate that the Boussinesq equation, born in 1872 as a model of the solitary wave, is today an indispensable tool for the protection of coastal communities [70-71].

The enhanced Boussinesq-type models are also applied to internal waves, where the two-layer and continuously stratified formulations lead to Boussinesq systems with improved dispersion [13,54]. The modelling of internal solitary waves in the coastal ocean, and of their generation by topography and their interaction with the mean flow, relies on the Boussinesq-type systems reviewed in Section 7 [56-57]. The strongly nonlinear Serre-Green-Naghdi equations, which generalize the Boussinesq theory beyond the weakly nonlinear regime, are the subject of an active literature on their well-posedness, solitary waves, and dispersive shock waves [72-75]. The Green-Naghdi theory for shallow water waves was developed in the survey of Webster, Duan, and Zhao [76]. Congy, El, Gavrilyuk, Hoefer, and Shyue studied the solitary wave-mean flow interaction in the Serre-Green-Naghdi equations using Whitham modulation theory, providing analytical descriptions of the trapping and transmission of solitary waves by slowly varying flows [77]. Kamchatnov constructed integrable dispersive generalizations of the Ovsyannikov two-layer shallow-water model, which provide periodic and soliton solutions of the corresponding Boussinesq-type systems [78]. Hao and Li established the ill-posedness and well-posedness results for the inviscid Boussinesq equations [79]. These developments connect the classical Boussinesq theory with the modern theory of strongly nonlinear dispersive wave propagation [73,77].

Exact and Explicit Solutions

The Boussinesq equation and its many variants admit a rich variety of exact solutions, which have been constructed by a broad range of analytical methods. Wazwaz constructed soliton solutions and periodic solutions of the Boussinesq equation by the modified decomposition method [80], and later derived multiple soliton solutions and multiple complex soliton solutions for two distinct Boussinesq equations [81]. Zhang constructed multi-soliton solutions of the Boussinesq equation using a modified Bäcklund transformation [82]. Yan constructed solitary-pattern solutions for the two-dimensional nonlinear dispersion Boussinesq equation [83]. Feng obtained traveling solitary-wave solutions to the generalized Boussinesq equation by direct integration [84]. These exact solutions provide benchmarks for the numerical methods and illuminate the structure of the solution families [80,82,84].

The Hirota bilinear method has been applied systematically to the Boussinesq equation and its generalizations. Hemnath, Saha, and Bhushan studied the resonant soliton, breather, and lump dynamics in the Boussinesq-type equation via the Hirota bilinear method, providing exact solutions with applications to geophysical waves [85]. Wazwaz constructed a variety of soliton solutions for the Boussinesq-Burgers equation and the higher-order Boussinesq-Burgers equation, including multiple-soliton and periodic solutions [86]. Khalfallah derived exact traveling wave solutions of the Boussinesq-Burgers equation [87]. Hon and Fan constructed exact solutions for the coupled Higgs field equation and the coupled Schrödinger-Boussinesq equation [88]. These bilinear and coupled-solution results connect the Boussinesq family to the broader theory of integrable systems and exact solution methods [81,85].

The logarithmic Boussinesq equation and its regularized variant admit Gaussian solitary waves, in analogy with the log-KdV and log-KP equations. Wazwaz derived these Gaussian solitary waves for the logarithmic Boussinesq equation and the logarithmic regularized Boussinesq equation, extending the exact-solution theory to the logarithmic nonlinearities [89]. The generalized improved Boussinesq equation was studied by Motsepa and Khalique, who derived exact solutions and conservation laws [90]. The soliton solutions of the generalized modified BBM equation and the generalized Boussinesq equation were derived by Güner by the ansatz method, providing dark and bright soliton solutions of the power-law models [91]. The ill-posed Boussinesq equation was shown by Yaşar, San, and Özkan to be nonlinearly self-adjoint, with conservation laws and exact solutions constructed by Lie symmetry methods and the exp-function method [42]. Tian performed a Lie symmetry analysis of a fourth-order nonlinear generalized

Boussinesq water wave equation, deriving conservation laws and solitary wave solutions [92]. Yang obtained new traveling wave solutions of the sixth-order Boussinesq equation by the tanh-coth method [93]. The comprehensive treatment of the Boussinesq equation and its solitary-wave theory in the monograph of Wazwaz provides the standard reference for the exact-solution methods [94]. These results demonstrate the breadth of the exact-solution literature for the Boussinesq family [89-90,92].

Fractional and Coupled Variants

Fractional generalizations of the Boussinesq equation have been studied extensively in recent years. The time-fractional good Boussinesq equation, with the Caputo fractional derivative, has been solved numerically by radial basis function collocation and quintic B-spline methods [65]. The conformable space-time fractional Boussinesq equation was solved exactly by Chen, Zhu, and Qi, who obtained Jacobian double periodic, simply periodic, and rational solutions by the F-expansion and (G'/G)-expansion methods [95]. Javeed, Saif, Waheed, and Baleanu derived exact solutions of the fractional mBBM equation and the coupled system of fractional Boussinesq-Burgers equations [96]. These fractional studies extend the classical theory of the Boussinesq family to anomalous diffusion and nonlocal wave propagation [95-96].

Coupled Boussinesq systems arise in several physical contexts. The coupled Higgs field equation and the coupled Schrödinger-Boussinesq equation were solved exactly by Hon and Fan [88]. The fractional coupled system of Boussinesq-Burgers equations was treated by Javeed and collaborators [96]. The KdV-KdV systems of Boussinesq type, which couple two KdV equations, were studied numerically by Bona, Dougalis, and Mitsotakis [52], and their asymptotic behavior was analyzed by Capistrano-Filho and Gallego [53]. The generalized cubic Boussinesq-type model for shallow-water wave dynamics was studied by Biswas and collaborators, who recovered the solitary waves, shock waves, cnoidal waves, and conservation laws of the model [97]. The Boussinesq-Burgers equations, which couple the Boussinesq equation with a Burgers-type equation, were solved exactly by Khalfallah and by Wazwaz [86-87].

Lattices and Discrete Origins

The Boussinesq equation arises as the continuum limit of the Fermi-Pasta-Ulam-Tsingou lattice, the one-dimensional chain of nonlinearly coupled masses whose recurrence phenomenon motivated much of modern nonlinear science [8]. The travelling-wave theory of the FPU lattice was developed by Iooss, who proved the existence of travelling waves with prescribed properties [6]. Schneider and Wayne proved that the counter-propagating waves on the FPU lattice are well approximated by the solutions of the continuum Boussinesq-type equations over long time scales [7]. These results establish the Boussinesq equation as the universal continuum model for bidirectional wave propagation in nonlinear lattices [6-7].

The lattice connection continues to generate new results. Nfor, Yamgoué, and Kakmeni investigated the bright and dark solitons in the alpha-beta Fermi-Pasta-Ulam lattice, deriving the extended KdV and NLS amplitude equations and studying the modulational instability of the system [20]. Kirane, Stalin, Arun, and Lakshmanan studied soliton molecules in the Fermi-Pasta-Ulam-Tsingou lattice using the Gardner equation approach, connecting the discrete lattice dynamics to the continuum Gardner equation [21]. These studies demonstrate that the Boussinesq-type equations remain at the interface of discrete and continuum nonlinear dynamics [20-21].

OPEN PROBLEMS AND FUTURE DIRECTIONS

Several fundamental questions about the Boussinesq equations remain open. First, the complete well-posedness theory of the classical (bad) Boussinesq equation, and the characterization of the sense in which its solutions are meaningful, deserves further development [11,42]. Second, the sharp description of the blow-up dynamics, including the precise rates and the classification of blow-up profiles for the multidimensional and paradigm equations, is not fully resolved [28,32]. Third, the rigorous derivation of the Boussinesq systems from the full water-wave and internal-wave problems, with sharp error estimates over physically relevant time scales, continues to be refined [4,19]. Fourth, the stability theory for the two-dimensional paradigm equation and for the Boussinesq systems in several space dimensions is less complete than the one-dimensional theory [31,39].

In applications, several directions stand out. The high-order Boussinesq-type models for coastal engineering, including their coupling to nonlinear shallow-water solvers and their application to wave breaking and run-up, continue to be developed and validated against laboratory and field data [67,70-71]. The strongly nonlinear Serre-Green-Naghdi equations and their extensions, including the interaction of solitary waves with mean flows and dispersive shock waves, are an active frontier [75,77]. The fractional and stochastic variants of the Boussinesq family, and their numerical solution with conservative schemes, offer a rich agenda for both theory and computation [65,95]. The data-driven modelling of shallow-water and internal waves, using physics-informed neural networks and related methods, is an emerging direction that leverages the exact solutions of the Boussinesq family as benchmarks [85,97].

From a computational perspective, the design of structure-preserving schemes that conserve all the invariants of the Boussinesq paradigm equation, and the extension of the conservative finite-difference and spectral methods to the fractional, stochastic, and higher-dimensional variants, are important goals [45-46]. The development of adaptive methods based on the a posteriori analysis of the conservative schemes will enable efficient computation near blow-up and in the presence of dispersive shock waves [32,62]. The rigorous numerical analysis of the two-dimensional paradigm equation, including the sharp convergence rates of the conservative and spectral methods, deserves further work [31,63]. Finally, the interplay between the Boussinesq theory, the discrete lattice

dynamics, and the modern theory of asymptotic water-wave models guarantees that the equation will continue to generate new problems at the interface of analysis, numerics, and physical oceanography [6,19,77].

CONCLUSION

The Boussinesq equation is one of the most consequential equations in the physical sciences. It was the first mathematical model to explain the solitary wave observed by Russell in 1834, and it founded the theory of nonlinear dispersive wave equations [1-2]. Its family of generalizations, from the good and bad Boussinesq equations to the Boussinesq systems, the paradigm equation, and the Boussinesq-type engineering models, has made it the canonical model for bidirectional, weakly nonlinear, weakly dispersive wave propagation across physics, engineering, and applied mathematics [13,66-67]. Its solitary-wave theory, well-posedness, and blow-up dynamics have served as a laboratory for the modern theory of nonlinear dispersive equations [9-10,28].

The trajectory of research on the Boussinesq equation mirrors the broader development of nonlinear science: a single equation, born from the attempt to explain a wave observed in a canal, grew into a universal model whose reach now extends from the mathematical theory of water waves to the engineering practice of coastal protection and tsunami hazard assessment [1,70-71]. The recent literature, including the work of 2025 and 2026 on strongly nonlinear shallow-water models, exact solutions of the Boussinesq-type equations, and numerical methods for the fractional variants, shows that the subject remains extraordinarily vibrant [65,77,85,97]. We expect that the interplay between analysis, computation, and physical observation will continue to drive progress on this remarkable family of equations for decades to come.

DECLARATIONS

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