

Analytical Solutions of Fractional Differential Equations: A Case Study Using MathHandbook

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ABSTRACT

This study investigates analytical solutions of fractional differential equations (FDEs) using the MathHandbook symbolic computation platform. The objective is to evaluate how the unified operator $ds(y, x, \alpha)$ and the dsolve engine handle differential equations of arbitrary order, including integer-order ODEs/PDEs, fractional ODEs and PDEs, fractional integral equations, mixed differential-integral equations, systems of fractional equations, complex-order equations, and variable-order equations. Comparative analysis shows that general solutions exhibit order-dependent exponential behavior, while particular solutions remain invariant for linear equations with identical forcing terms. These results highlight the coherence of fractional calculus and demonstrate how AI-driven symbolic computation enhances mathematical modeling, verification, and mathematics education.

Keywords: Fractional Differential Equations; Fractional Calculus; Symbolic Computation; Mathhandbook; Analytical Solutions; Differential Equations; Fractional-Order Derivatives; Variable-Order Equations; Complex-Order Equations; AI-Driven Mathematics

INTRODUCTION

Differential equations form the foundation of mathematical physics, modeling phenomena such as heat conduction, diffusion, elasticity, gravitation, and electrochemistry [1-6]. While classical differential equations involve integer-order derivatives, many real-world systems exhibit memory, nonlocality, and hereditary behavior that are better captured by fractional differential equations (FDEs) [7-9].

Fractional calculus extends differentiation and integration to arbitrary real or complex orders. FDEs appear in physics, chemistry, biology, biophysics, economics, control theory, and signal processing. Nonlinear fractional models often display rich dynamical behavior, including chaos [10-11].

This case study evaluates MathHandbook—an evolution of SymbMath—as a platform [12-15] for solving fractional and classical differential equations analytically.

The MathHandbook Platform

MathHandbook is an online symbolic mathematics engine capable of solving differential equations of arbitrary order. Its dsolve function supports:

- Integer-order odes and pdes
- Fractional differential equations (Caputo definition by default)
- Fractional integral equations
- Mixed differential-integral equations
- Systems of fractional equations
- Complex-order differential equations
- Variable-order differential equations

Users may input equations directly (e.g., $y(1,x)-2y=\exp(x)$) or use symbolic operators such as $ds(y, x, \alpha)$ for fractional derivatives.

Methodology

Unified Differential Operators

MathHandbook uses a unified operator

$$ds(y, x, \alpha) = \frac{d^\alpha y}{dx^\alpha}, \quad \alpha \in \mathbb{C}.$$

to represent derivatives and integrals of arbitrary order:

$$\begin{aligned} ds(y, x, n) &= \frac{d^n y}{dx^n}, & n > 0, \\ ds(y, x, -n) &= \int^n y \, dx, & n > 0, \end{aligned}$$

Caputo derivative (default):

$${}^c D_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha} f'(t) \, dt, \quad 0 < \alpha < 1$$

where Γ is the Gamma function.

Riemann–Liouville definition:

$${}^{RL} D_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^x (x-t)^{-\alpha} f(t) \, dt.$$

Key identity:

$$\frac{d^\alpha}{dx^\alpha} e^x = e^x, \quad \text{for all } \alpha \in \mathbb{C}.$$

Since e^x is an eigenfunction of both the Caputo and Riemann–Liouville operators, exponential terms retain their structure across all orders. This property ensures that exponential terms appearing in general solutions retain their structure across all differential orders, which is essential for the unified analysis presented in later sections.

Example Input

Consider the first-order linear differential equation:

$$\frac{dy}{dx} - 2y = e^x.$$

MathHandbook input:

`dsolve (ds(y,x,1) - 2*y = exp(x) )`

The system returns both the general solution and a particular solution: $C_1 e^{2x} - e^x$.

General solution: $y_g = C_1 e^{2x}$

Particular solution: $y_p = -e^x$

Differential Operators Across Orders

Table 1 summarizes the operator notation for a range of derivative and integral orders, including fractional, negative, complex, and variable orders. This provides a unified framework for expressing all examples used in Results.

Order	Equation	Operator Form
2	$\frac{d^2 y}{dx^2} - 2y = e^x$	$ds(y, x, 2)$
1.5	$\frac{d^{1.5} y}{dx^{1.5}} - 2y = e^x$	$ds(y, x, 1.5)$
1	$\frac{dy}{dx} - 2y = e^x$	$ds(y, x)$
0.5	$\frac{d^{0.5} y}{dx^{0.5}} - 2y = e^x$	$ds(y, x, 0.5)$
-0.5	$\frac{d^{-0.5} y}{dx^{-0.5}} - 2y = e^x$	$ds(y, x, -0.5)$
-1	$\int y dx - 2y = e^x$	$ds(y, x, -1)$
-2	$\iint y (dx)^2 - 2y = e^x$	$ds(y, x, -2)$
i	$\frac{d^i y}{dx^i} - 2y = e^x$	$ds(y, x, i)$
$\cos(x)$	$\frac{d^{\cos(x)} y}{dx^{\cos(x)}} - 2y = e^x$	$ds(y, x, \cos(x))$

Table 1: Differential Operator Notation Across Different Orders

RESULTS

This section presents a comprehensive set of analytical results obtained using the *MathHandbook* dsolve engine. In addition to the comparison of differential orders (Table 1), we include a detailed comparison of general and particular solutions across integer, fractional, integral, complex, and variable orders (Table 2). We then extend the results to cover ordinary differential equations, partial differential equations, fractional PDEs, integral equations, mixed-order equations, systems of fractional equations, complex-order equations, and variable-order equations.

Ordinary Differential Equations (ODEs)

MathHandbook solves both symbolically and numerically solvable ODEs.

Example:

$$y' = \exp(y)$$

This first-order nonlinear ODE is solved:

$$y = -\log(-C_1 - x).$$

Partial Differential Equations (PDEs)

MathHandbook supports 2D and 3D visualization of PDE solutions. Classical PDEs such as Laplace, Poisson, diffusion, and wave equations are included.

Examples:

- Laplace equation: $\Delta y = 0$
- Poisson equation: $\Delta y = \rho$
- Heat equation: $y_t - a^2 \Delta y = f(t, x)$
- Wave equation: $y_{tt} - a^2 \Delta y = f(t, x, z)$

Solutions can be visualized using plot2D and plot3D, enabling interactive exploration of time-dependent behavior.

Fractional Partial Differential Equations

Fractional PDEs are obtained by replacing the time derivative with a fractional order α , typically $0 < \alpha < 1$.

Example:

$$\frac{d^{0.5}y}{dt^{0.5}} = y_x - e^x - e^t$$

MathHandbook solves such equations symbolically when possible, or numerically when closed-form solutions are unavailable.

Integral and Fractional Integral Equations

Negative-order derivatives treated as integrals. Fractional integral equations extend this to non-integer negative orders.

Examples:

$$\int y dx - 2y = e^x$$

$$\frac{d^{-0.5}y}{dx^{-0.5}} - 2y = e^x$$

MathHandbook converts these to differential form by differentiation, solves them, and then verifies the solution.

Mixed Differential-Integral Order Equations

Equations combining fractional derivatives and fractional integrals are also supported.

Example:

$$\frac{d^\pi y}{dx^\pi} - \int y(x) dx^\pi = e^x$$

Systems of Fractional Differential Equations

Systems of equations are solved simultaneously.

Example:

$$\frac{d^{0.5}x}{dt^{0.5}} = 2x, \quad \frac{d^{0.5}y}{dt^{0.5}} = x + y$$

Complex-Order Differential Equations

MathHandbook extends differentiation to complex orders, producing oscillatory-exponential solutions.

Example:

$$\frac{d^i y}{dx^i} - 2y = e^x$$

The general solution involves $e^{\frac{1}{2}ix}$, reflecting complex-order behavior.

Variable-Order Differential Equations

When the order varies with the independent variable, the solution typically involves Mittag-Leffler functions.

Example:

$$\frac{d^{\cos(x)} y}{dx^{\cos(x)}} - 2y = e^x$$

The general solution is:

$$y_g(x) = C_1 E_{\cos(x)}(2x^{\cos(x)})$$

where $E_\alpha(\cdot)$ is the Mittag-Leffler function.

Order	Equation	General Solution	Particular Solution
2	$\frac{d^2 y}{dx^2} - 2y = e^x$	$C_1 e^{\sqrt{2}x}$	$-e^x$
1.5	$\frac{d^{1.5} y}{dx^{1.5}} - 2y = e^x$	$C_1 e^{\frac{2}{\sqrt{2}}x}$	$-e^x$
1	$\frac{dy}{dx} - 2y = e^x$	$C_1 e^{2x}$	$-e^x$
0.5	$\frac{d^{0.5} y}{dx^{0.5}} - 2y = e^x$	$C_1 e^{4x}$	$-e^x$
-0.5	$\frac{d^{-0.5} y}{dx^{-0.5}} - 2y = e^x$	$C_1 e^{x/4}$	$-e^x$
-1	$\int y dx - 2y = e^x$	$C_1 e^{x/2}$	$-e^x$
-2	$\iint y (dx)^2 - 2y = e^x$	$C_1 e^{x/\sqrt{2}}$	$-e^x$
i	$\frac{d^i y}{dx^i} - 2y = e^x$	$C_1 e^{\frac{i}{2}x}$	$-e^x$
$\cos(x)$	$\frac{d^{\cos(x)} y}{dx^{\cos(x)}} - 2y = e^x$	$C_1 E_{\cos(x)}(2x^{\cos(x)})$	$-e^x$

Table 2: Summary of All Examples

DISCUSSION

The results of this study reveal several structural properties that are fundamental to understanding fractional differential equations and their computational treatment. First, the preservation of linear structure across fractional orders demonstrates that fractional derivatives extend classical calculus in a mathematically coherent way.

Table 2 shows that solution structures of differential equations are the same regardless of its order:

$$\text{solution} = \text{general solution} + \text{particular solution} = \text{gsolution}() + \text{psolution}()$$

The smooth variation of the general solution with respect to the order α highlights the continuous transition between integer-order and fractional-order dynamics, offering a unified framework for modeling systems with memory or hereditary effects.

A particularly notable finding is the invariance of the particular solution across all constant orders for the equation

$$d^\alpha y/dx^\alpha - 2y = e^x,$$

which underscores the robustness of forcing-term responses in linear fractional systems. This behavior simplifies analytical interpretation and provides a stable foundation for numerical verification.

The emergence of Mittag-Leffler functions in variable-order equations reflects the richer dynamical landscape introduced by non-constant differentiation orders. These functions generalize exponential behavior and naturally encode memory effects, making them indispensable in modeling anomalous diffusion, viscoelasticity, and other nonlocal processes. Similarly, complex-order derivatives produce oscillatory-exponential hybrid solutions, suggesting potential applications in signal processing, wave propagation, and systems exhibiting combined growth and rotation.

Overall, the results demonstrate that fractional calculus offers a flexible and powerful modeling paradigm. The ability of MathHandbook to compute symbolic solutions across such a wide range of orders including fractional, integral, mixed, complex, and variable orders illustrates the maturity of modern symbolic computation and its value for both theoretical research and applied problem solving.

Educational Implications

AI-powered symbolic computation tools such as MathHandbook significantly enhance mathematics education by:

- providing instant analytical solutions
 - supporting experimentation with fractional orders
 - visualizing PDEs in 2D and 3D
 - enabling students to verify solutions interactively
 - bridging theory and computation in differential equations courses
- As AI continues to evolve, such tools will play an increasingly central role in personalized and exploratory learning.

CONCLUSION

This case study provides a comprehensive examination of fractional differential equations solved using the MathHandbook symbolic computation platform. The system successfully handles an extensive spectrum of differential operators, including integer-order, fractional-order, integral, mixed-order, complex-order, and variable-order derivatives. The comparative analysis across orders reveals consistent structural patterns: general solutions exhibit order-dependent exponential behavior, while particular solutions remain invariant for linear equations with identical forcing terms.

These findings reinforce the theoretical coherence of fractional calculus and highlight its capacity to model memory-dependent and nonlocal phenomena. The integration of AI-driven symbolic computation into MathHandbook further enhances accessibility, enabling researchers, educators, and students to explore fractional models interactively and verify solutions analytically or numerically.

As fractional calculus continues to expand across physics, engineering, biology, and applied mathematics, tools such as MathHandbook will play an increasingly important role in supporting research, facilitating experimentation, and bridging the gap between theory and computation. The results presented here demonstrate that modern symbolic engines are not only capable of solving complex fractional systems but also of providing insights that deepen our understanding of their underlying mathematical structure.

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