

The Klein–Gordon Equation: From Relativistic Quantum Mechanics to Nonlinear Field Dynamics - A Comprehensive Review

Weiguang Huang*

Department of Chemistry, University of New South Wales, Sydney, Australia

*Correspondence to: Weiguang Huang, Department of Chemistry, University of New South Wales, Sydney, Australia, E-mail: drhuangau@yahoo.com.au

Received: August 23, 2026; Manuscript No: JMPM-26-6513; Editor Assigned: August 27, 2026; PreQc No: JMPM-26-6513 (PQ); Reviewed: September 09, 2026; Revised: September 14, 2026; Manuscript No: JMPM-26-6513 (R); Published: October 06, 2026

Citation: Huang W (2026). The Klein–Gordon Equation: From Relativistic Quantum Mechanics to Nonlinear Field Dynamics -A Comprehensive Review. J. Math. Phys. Mech. Vol.1 Iss.1, October (2026), pp:187-196.

Copyright: © 2026 Weiguang Huang. This is an open-access article distributed under the terms of the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original author and source are credited.

ABSTRACT

The Klein–Gordon equation (KGE), formulated in 1926 as the first relativistic wave equation, remains one of the most broadly consequential equations in theoretical physics. This review surveys the KGE across four interconnected dimensions: its historical genesis from de Broglie’s matter-wave hypothesis and five-dimensional relativity; the interpretational crisis of negative probability densities that motivated second quantization and the modern quantum-field-theoretic viewpoint; the rich mathematical theory of linear and nonlinear variants, including well-posedness, Strichartz estimates, solitary waves, stability theory, and breather nonexistence; and the expanding landscape of numerical and computational methods, from exponential integrators and filtered finite differences to physics-informed neural networks and quantum simulation algorithms. Particular attention is paid to recent developments (2020–2026), including codimension-one stability results, dispersive estimates on cones and inverse-square potentials, long-lived quantum oscillons, and machine-learning solvers for hyperbolic systems. Application domains are reviewed in depth: coherent scalar-field dynamics in cosmology, axion dark matter and oscillon formation, sine-Gordon phenomenology in Josephson junctions and quasi-one-dimensional conductors, Klein tunneling in graphene, and superradiant instabilities of massive scalar fields around rotating black holes. Open problems are identified at the interface of rigorous analysis, high-performance simulation, and data-driven modeling. This review is intended both as a self-contained introduction for graduate students and as a reference map of the current research frontier.

Keywords: Klein–Gordon Equation; Relativistic Wave Equation; Nonlinear Dispersive Equations; Solitons and Oscillons; Scalar Field Cosmology; Physics-Informed Neural Networks; Exponential Integrators; Quantum Simulation

INTRODUCTION

Few equations in theoretical physics have displayed the longevity and cross-disciplinary reach of the Klein–Gordon equation. Formulated within months of Schrödinger’s nonrelativistic wave mechanics indeed, arguably before it in its relativistic form and applied immediately to the Compton effect, the equation emerged from de Broglie’s matter-wave hypothesis as the natural quantum evolution law for any free particle respecting special relativity. A century later, it underlies the Standard Model description of scalar degrees of freedom, guides the phenomenology of axion dark matter, governs the collective modes of superconducting circuits, and serves as a canonical benchmark for new computational paradigms [1-3].

The term “quantum” is used in this review exclusively in the sense of quantum mechanics the physics of quantized energy levels, wave functions, and their field-theoretic extension and not in the sense of quantum calculus (q-calculus), the parallel mathematical theory of q-deformed difference operators and their fractional and multiplicative generalizations. The two subjects share a name but little else: quantum calculus concerns the q-analogue of the derivative, $D_q f(x) = (f(qx) - f(x))/((q - 1)x)$, together with the associated integrals and inequalities, and has developed into a substantial branch of approximation theory and mathematical inequalities [65][66][67], whereas the present review concerns the relativistic wave equation and its quantization. To avoid ambiguity, we retain “quantum” for the physical theory throughout, and we refer the reader interested in the q-calculus literature to recent studies of dual basic symmetric quantum calculus, (p, q)-symmetric integral inequalities [66], and quantum multiplicative calculus [4-5].

The equation’s importance stems from its position at the intersection of three deep structures. First, it is the simplest linear hyperbolic equation with a massive dispersion relation, making it the prototypical laboratory for relativistic quantum mechanics: every subtlety of antiparticles, pair creation, and causality appears already at this level [4]. Second, its nonlinear generalizations with polynomial, sine, or more exotic potentials are among the most intensively studied nonlinear dispersive equations in

mathematical analysis. They exhibit solitary waves, breathers, blow-up, scattering, and modified-scattering phenomena whose rigorous understanding has driven decades of progress in PDE theory. Third, the equation is experimentally operational far beyond high-energy physics. The chiral charge carriers of graphene obey a massless Dirac equation, but bilayer systems and certain tunneling geometries realize Klein–Gordon-type kinematics with measurable Klein paradox phenomenology [5].

Nonlinear Klein–Gordon dynamics dominate modern applications. Oscillons long-lived, localized, oscillating field configurations supported by scalar potentials have become central to models of post-inflationary reheating and structure formation, while axion-like fields, whose cosmological evolution is governed by a Klein–Gordon equation with periodic potential, are leading dark-matter candidates [6-7]. On the computational side, the equation has become a proving ground for methods that then transfer across scientific computing: exponential integrators exploit its exactly solvable linear part for orders of magnitude gains over naive time steppers; physics-informed neural networks (PINNs) use it as the archetypal second-order-in-time hyperbolic test problem; and quantum algorithms for scalar field theories were among the first demonstrations that quantum computers can simulate continuum QFT dynamics [8-10].

The case for a fresh synthesis is timely. On the analytical side, the past five years have delivered decisive results long sought: codimension-one stability of focusing solitons, modified scattering for quadratic nonlinearities, and sharp Strichartz theory in higher dimensions [11-13]. On the computational side, machine-learning solvers have matured from proofs of concept to methods with error analysis for second-order hyperbolic problems and applications as demanding as critical gravitational collapse [14-15]. Observationally, ultralight scalar bosons are now constrained by black-hole spin measurements and pursued by dedicated haloscope experiments, turning the equation’s solutions into testable predictions [16-17]. A review that connects these threads serves analysts, computational scientists, and phenomenologists simultaneously.

This review synthesizes these threads into a single narrative. Section 2 describes the literature search and the criteria used to select the works surveyed. Section 3 traces the historical genesis of the equation and the interpretational crisis of 1926–1930. Section 4 describes the passage from single-particle interpretation to second quantization. Section 5 surveys the mathematical theory of linear and nonlinear KGEs, emphasizing results of the past decade. Section 6 reviews numerical methods from classical integrators to machine learning and quantum simulation. Section 7 examines physical applications across cosmology, condensed matter, and gravitation. Section 8 identifies open problems, and Section 9 concludes.

REVIEW CONTENT

This review was compiled from a systematic search of the literature on the Klein–Gordon equation and its relatives. The search was conducted in the bibliographic databases Web of Science, Scopus, MathSciNet, zbMATH, INSPIRE-HEP, and NASA ADS, with Google Scholar and Semantic Scholar used for forward citation chasing and the arXiv preprint server (sections hep-th, hep-ph, math.AP, and nlin.PS) consulted for the most recent and forthcoming work. The reference lists of the retrieved articles, and of the standard monographs and review articles on relativistic wave equations, quantum field theory, and scalar-field cosmology, were examined in turn, so that a backward snowballing step could recover earlier or more specialized contributions that the keyword search alone did not return.

The search strings combined the terms “Klein–Gordon” and “Klein–Gordon equation” with one or more of: relativistic wave equation, second quantization, scalar field, nonlinear Klein–Gordon, sine–Gordon, solitary wave, soliton stability, Strichartz estimates, blow-up, breather, oscillon, axion, dark matter, superradiance, exponential integrator, physics-informed neural network, and quantum simulation. Boolean operators were used to combine the terms, and truncation was applied where the database syntax permitted it.

No lower limit was placed on the publication date, so that the historical development of the subject could be represented; the earliest work retained is from 1925 and the most recent from 2026. Particular attention was paid to the literature of the past decade, and works accepted or in press at the time of writing were included when a digital object identifier was available. A study was included if it introduced, derived, analysed, or applied the Klein–Gordon equation or one of its variants and generalizations, or if it reported a mathematical result, a numerical method, or a physical application directly relevant to that family; peer-reviewed journal articles, monographs, and conference proceedings were prioritized. A study was excluded if the Klein–Gordon equation appeared only as a passing comparison, if the full text could not be obtained, or if the work was a duplicate or a preliminary version superseded by a later publication.

The retained works were then grouped by theme historical genesis and interpretation, second quantization, the mathematical theory of linear and nonlinear equations, numerical and computational methods, and physical applications across cosmology, condensed matter, and gravitation and each section of the review corresponds to one of these themes. The review is a critical synthesis rather than a quantitative meta-analysis, because the literature is heterogeneous in method and scope.

Historical Origins and Early Interpretations

Matter Waves and the Relativity Problem

The conceptual chain leading to the KGE began with Louis de Broglie’s 1925 doctoral thesis, which proposed that material particles possess an intrinsic periodicity a wave associated with momentum p and wavelength $\lambda = h/p$ [3]. De Broglie himself recognized that

any such wave theory had to be compatible with Einstein's special relativity, and his notion of a "pilot wave" phase carried along by the particle implicitly contained a dispersion relation of exactly the form later derived by Klein and Gordon. The task of constructing the corresponding wave equation fell to the next generation.

In early 1926 several authors independently arrived at the same equation through different routes, a striking example of convergent discovery. Klein derived it by embedding quantum dynamics in a five-dimensional spacetime, unifying the electromagnetic four-potential with a compact fifth dimension [1]. Gordon obtained the same operator when computing Compton scattering amplitudes within Schrödinger's wave mechanics, demonstrating that the equation correctly reproduces the relativistic kinematics of the effect [2]. The historical literature sometimes credits Fock, Schrödinger, and de Broglie as co-discoverers as well; the name "Klein-Gordon equation" has nonetheless persisted, and we adopt it here.

Mathematically, the construction follows a template still taught today: start from the relativistic energy-momentum relation $E^2 = p^2c^2 + m^2c^4$ and promote $E \rightarrow i\hbar\partial_t$, $p \rightarrow -i\hbar\nabla$. The resulting equation is second order in time a structural departure from Schrödinger's first-order equation with far-reaching consequences. Initial data must specify both field and time derivative, the conserved current involves time derivatives of the field rather than the density alone, and the superposition of positive- and negative-frequency branches becomes unavoidable. Klein's five-dimensional derivation additionally embedded electromagnetism into the formalism through minimal coupling, anticipating gauge-covariant generalizations that would become routine in quantum field theory [1,4]. Gordon's Compton calculation demonstrated within months that the new equation reproduced relativistic kinematics correctly at the amplitude level, establishing its physical credibility independent of interpretational worries [2].

The Interpretational Crisis

The equation's early promise collapsed on contact with interpretation. For a complex scalar wavefunction ψ , the conserved current implied by the KGE has a density ρ proportional to $\text{Im}(\psi^*\partial_t\psi)$ rather than $|\psi|^2$, and this quantity is not positive definite. Superpositions of positive- and negative-frequency modes therefore produce states with negative total "probability," which is physically unacceptable for a one-particle theory. The difficulty was sharpened dramatically by Oskar Klein's analysis of scattering off a steep potential step [18]. In what is now called the Klein paradox, an incoming electron incident on a sufficiently high barrier is reflected with transmission properties impossible for a single particle: the step appears to generate particle-antiparticle pairs, though the language of pair creation did not yet exist. Klein's paradox exposed not a defect of the equation but a defect of forcing it into a single-particle straightjacket an insight whose full significance would take three more decades to crystallize.

Dirac's 1928 equation offered an elegant alternative for electrons, with a positive-definite density and spin emerging naturally [19]. For two decades the Dirac equation dominated relativistic quantum mechanics, and the KGE was often treated as a historical curiosity appropriate only for spinless mesons. Standard references of the mid-century era codified both equations side by side, cataloguing the pathologies of each in first quantization [20].

Rehabilitation and Modern Status

Three developments restored the KGE to foundational status. First, the identification of pions as spin-zero mesons gave the equation direct phenomenological content, since pionic wavefunctions satisfy Klein-Gordon kinematics. Second, Feshbach and Villars showed that the equation admits a Schrödinger-like two-component formulation in which position and density acquire a consistent (though still pair-producing) interpretation [21]. Third, and decisively, renormalized quantum field theory reframed negative-frequency modes as antiparticles and negative densities as symptoms of particle nonconservation rather than defective probabilities [4]. Within QFT the equation is simply the Euler-Lagrange equation of the free real or complex scalar field the starting point for perturbation theory, lattice simulation, and essentially every model-building exercise in particle physics.

The equation also acquired unexpected geometrical significance. In Kaluza-Klein reductions of higher-dimensional gravity, the components of the metric along compact directions behave precisely as Klein-Gordon fields, linking the equation to unified field theories [22]. And in the same year that the equation was born, the mechanism that would make scalar fields central to mass generation was foreshadowed: the 1964 papers on spontaneous symmetry breaking showed that a scalar field with a Mexican-hat potential, quantized via its Klein-Gordon kinetic term, can generate particle masses while preserving gauge invariance [23-25]. Every Higgs-sector calculation thus descends directly from the 1926 equation.

From Single Particle to Fields: Interpretation and Second Quantization

Pathologies of First Quantization

$$\rho = \frac{i\hbar}{2mc^2} \left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right), \quad \mathbf{j} = -\frac{i\hbar}{2m} (\psi^* \nabla \psi - \psi \nabla \psi^*)$$

In its first-quantized form the KGE assigns a complex wavefunction $\psi(\mathbf{x},t)$ to a single spinless particle. The conserved current derived from the equation takes the form

and ρ , unlike $|\psi|^2$ in Schrödinger mechanics, is indefinite. A wavepacket built from both positive- and negative-frequency components can carry zero or negative total density even though its energy is positive. Localization compounds the problem: confining a particle to a region smaller than its Compton wavelength $\lambda = \hbar/mc$ necessarily populates negative-energy states, so no position operator exists with probability-positive eigenstates. These features, catalogued exhaustively in classic texts, are not mathematical artifacts but physical signals that particle number is not conserved when energies reach the pair-production threshold mc^2 [20].

Feshbach and Villars clarified the situation by transforming the equation into a two-component first-order-in-time system resembling the Schrödinger form, with the density becoming $\rho = |\varphi_1|^2 - |\varphi_2|^2$ [21]. In this representation positive- and negative-frequency sectors separate cleanly, the nonrelativistic limit emerges systematically, and the source of indefiniteness is manifest as the relative sign of two positive quantities. Their formulation remains the standard starting point for discussions of localization and measurement in relativistic quantum mechanics.

Second Quantization and Antiparticles

The definitive resolution reinterprets the field itself as the fundamental object. Upon second quantization, ψ becomes a field operator expanded in creation and annihilation operators; negative-frequency solutions multiply creation operators and describe antiparticles [4]. The conserved density becomes a charge density rather than a probability density, and its indefiniteness merely reflects the possibility of creating particle–antiparticle pairs. All “paradoxes” dissolve: Klein’s potential step genuinely produces pairs when the voltage exceeds $2mc^2$, exactly as the field-theoretic reading predicts. The conceptual price is a shift in what is being described states of a field on spacetime rather than trajectories of particles but the payoff is a framework in which causality, locality, and multiparticle processes coexist consistently, and in which the single-particle equation survives as the one-particle sector of Fock space [4]. Free scalar QFT is also computationally transparent the absence of spin and of interaction vertices in the free theory makes it the canonical example in every textbook treatment of Wick’s theorem and propagators [20][4] and Gordon’s original Compton calculation anticipated this consistency between relativistic kinematics and quantum amplitudes [2].

The scalar theory’s simplicity has made it the first target for quantitative many-body and field-theoretic computation on new platforms. Jordan, Lee, and Preskill constructed efficient quantum algorithms for scattering in φ^4 theory, computing S-matrix elements for a continuum interacting QFT on a quantum computer with cost polynomial in relevant parameters; the φ^2 sector they build upon is precisely the quantized Klein–Gordon field. Related lattice constructions simulate gauge-scalar systems directly on quantum hardware via orbifold decompositions [10,26].

The free quantized theory is exactly solvable: normal-ordering reduces all correlation functions to Wick contractions against the Feynman propagator $\Delta(x-y)$, the vacuum expectation $\langle 0 | T \varphi(x) \varphi(y) | 0 \rangle$ obtained by contour integration over the relativistic propagator poles. This solvability is not merely pedagogical it is the computational engine of interacting theory. Perturbation series express scattering amplitudes through propagators and vertices; Euclidean lattice formulations integrate the free Gaussian measure explicitly and sample interactions by Monte Carlo; and quantum algorithms exploit the closed-form linear evolution as the subroutine around which interaction terms are Trotterized [4,10]. Whenever a new computational platform appears, scalar field theory is its first benchmark precisely because this Klein–Gordon backbone is analytically known and numerically checkable.

Quantum Effects in Nonlinear Configurations

Second quantization also enriches the nonlinear configurations discussed later in this review. Classical oscillons radiate exponentially slowly; whether they persist once quantum fluctuations are included is a physically decisive question. Hertzberg computed the quantum radiation rate of oscillons and found it parametrically suppressed, supporting their long-lived status in a quantum theory [27]. More recently, Evslin, Romańczukiewicz, and Wereszczyński argued through a combination of analytic estimates and lattice evidence that quantum corrections may render oscillons effectively eternal, with implications for primordial black hole avoidance and relic abundances [28]. The dialogue between classical Klein–Gordon dynamics and its quantized counterpart remains an active research frontier.

Mathematical Theory of Linear and Nonlinear Klein–Gordon Equations

The Linear Equation: Well-Posedness, Dispersion, and Estimates

The free KGE in natural units reads

$$\square \phi + m^2 \phi = 0, \quad \square = \partial_t^2 - \Delta,$$

with dispersion relation $\omega(k) = \sqrt{|k|^2 + m^2}$. The Cauchy problem is well posed in the energy space $H^1 \times L^2$ for data (ϕ_0, ϕ_1) ; the conserved energy

$$E(t) = \frac{1}{2} \int (|\partial_t \phi|^2 + |\nabla \phi|^2 + m^2 |\phi|^2) dx$$

controls the solution globally. Unlike the wave equation, the mass term prevents decay of the L^2 norm, and pointwise decay proceeds at a modified rate $|t|^{-(d/2)}$ with an oscillatory phase slow enough to make nonlinear interactions delicate, yet fast enough to permit global small-data results. The systematic machinery for capturing this decay is the family of Strichartz estimates. For flat-space equations these are classical, but their extension to non-Euclidean settings remains active: Beceanu and Chen established Strichartz estimates for the KGE in three and higher dimensions with sharp derivative losses, consolidating the endpoint theory [13]. External potentials introduce further subtleties. Lee, Seo, and Seok derived local smoothing and Strichartz estimates for KGEs perturbed by critical inverse-square potentials, identifying the range of coupling constants for which dispersive control survives [29]. Geometry also matters: Yin and Zhang proved decay and Strichartz estimates for the massive equation on a cone—a model spacetime relevant to cosmic strings showing that the vertex singularity imposes additional integrability restrictions on admissible pairs [30].

Stability theory for coherent structures has advanced decisively in recent years. Nakanishi and Schlag developed the invariant-manifold framework describing the threshold between dispersion and blow-up for focusing equations, and its programmatic completion continues: Lührmann and Schlag established codimension-one orbital stability of the soliton for the one-dimensional

focusing cubic KGE, resolving a benchmark problem in the stability of topologically trivial coherent structures [11,31]. Comprehensive treatments of the geometric-analytic toolbox are collected in the monographs of Shatah and Struwe [32].

A useful organizing principle distinguishes energy-subcritical, -critical, and -supercritical nonlinearities by how the scaling symmetry of the equation acts on the conserved energy. Subcritical problems admit global theories via perturbative estimates; critical problems require concentration compactness or interaction-morawetz machinery to preclude energy concentration [33]; supercritical regimes remain largely open except under symmetry reductions. The invariant-manifold decomposition of Nakanishi and Schlag unifies this landscape near threshold: solutions either scatter or concentrate onto a ground state, with a co-dimension-one manifold separating the behaviors [31].

Nonlinear Equations: Existence, Stability, and Scattering

Nonlinear Kges Take the Form

$$\square\phi + V'(\phi) = 0, \quad V(\phi) \sim \frac{\lambda}{4}\phi^4 \text{ or } m^2/\beta^2 (1 - \cos\beta\phi),$$

where the sign and shape of V determine the phenomenology. Derrick's classic scaling argument showed that no time-independent, finite-energy, nontrivial solutions exist for scalar theories in more than one spatial dimension without additional stabilizing terms a no-go theorem that redirects attention to time-dependent localized states. Strauss demonstrated the existence of solitary waves in higher dimensions for a broad class of potentials [34-35]. Their stability is subtle: Shatah constructed stable standing waves via a variational/constraint analysis and exhibited unstable ground states whose instability manifests along symmetry-breaking directions [36-37]. His normal-form method showed that quadratic nonlinearities generically destroy small-amplitude periodic orbits through resonant energy transfer, reshaping how analysts treat low-order nonlinearities [38].

Global existence for small data is the cornerstone scattering result. Klainerman's seminal proof established global existence and scattering for small-amplitude solutions of semilinear KGEs in four spacetime dimensions, exploiting the favorable null structure of the mass term [39]. Ginibre and Velo extended the Cauchy-problem theory to general data in the energy space, proving global well-posedness and scattering for subcritical nonlinearities in the full expected range [40-41]. Modern refinements continue: Miao and Zheng settled energy scattering for the three-dimensional KGE with Hartree-type cubic convolution nonlinearity, an intercritical case requiring concentration-compactness alongside vector-field methods [33]. In one dimension with quadratic nonlinearity and a potential, Germain and Pusateri proved modified scattering the solution scatters to a nonlinear phase-corrupted asymptote rather than a free solution illustrating the logarithmic corrections characteristic of quadratic systems [12].

Breathers time-periodic localized solutions are the sine-Gordon equation's celebrated exact modes, but they are fragile. Segur and Kruskal proved that ϕ^4 theory admits no small-amplitude breathers, radiative tails being nonperturbative in the amplitude, and Denzler extended nonpersistence to perturbed sine-Gordon models [42-43]. Analytic descriptions of breather internal dynamics and interactions continue to be refined, sustaining interest in quasi-breathers supported by more exotic potentials [44].

Variational Methods and Threshold Phenomena

Standing waves arise as critical points of the conserved energy restricted to fixed charge or L^2 -norm constraints; the resulting Nehari-type variational problems yield ground states whose orbital stability follows from the constrained second variation [35][36]. Derrick's virial identity provides the complementary exclusion mechanism, converting scaling invariance into an algebraic obstruction for static configurations [34]. Together these tools delineate exactly which potentials support stable coherent structures. Below the ground-state energy threshold the dynamics is fully understood in many cases: small data scatter globally [39], while above threshold concentration onto the soliton or blow-up occurs. The recent codimension-one stability result completes this threshold picture for the one-dimensional focusing cubic model by constructing the center-stable manifold of the soliton, confirming that instability is confined to a thin manifold of initial conditions [11]. The interplay between variational structure and dynamical stability remains among the most productive interfaces in the modern theory [32].

Relations to the Schrödinger, Wave, and Dirac Equations

The Klein-Gordon equation is best understood as one member of a family of wave equations that share a common origin in the relativistic energy-momentum relation but differ in order, spin content, and the regime in which they apply. The nonrelativistic Schrödinger equation is recovered as the small-velocity limit: writing $\phi = e^{-imc^2 t/\hbar}\psi$ and letting $c \rightarrow \infty$ leaves the free Schrödinger equation $i\hbar\partial_t\psi = -(\hbar^2/2m)\Delta\psi$, so that the Klein-Gordon equation reduces to the Schrödinger equation up to the rest-mass phase. In the opposite limit $m \rightarrow 0$ the mass term vanishes and the Klein-Gordon equation becomes the classical wave equation $\square\phi = 0$, whose solutions propagate at the speed of light; the Klein-Gordon dispersion relation $\omega^2 = |k|^2 + m^2$ interpolates continuously between the nonrelativistic parabolic law and the massless linear law.

Dirac's equation arises from factoring the Klein-Gordon operator. Because the Klein-Gordon equation is second order in time and space, it admits no positive-definite conserved density in first quantization; Dirac therefore sought a first-order equation $(i\gamma^\mu\partial_\mu - m)\psi = 0$ whose square reproduces the Klein-Gordon operator. This factorization is possible only if the coefficients γ^μ are matrices, which is why the Dirac equation necessarily describes spin-1/2 particles and predicts antiparticles through its negative-energy solutions. The Klein-Gordon equation, by contrast, describes spin-0 fields; it remains second order, and its two-component Feshbach-Villars form plays the role for spinless particles that the Dirac spinor plays for spin-1/2 particles. In short, the wave equation is the massless, spinless case; the Schrödinger equation is the nonrelativistic limit; the Klein-Gordon equation is the

relativistic scalar case; and the Dirac equation is its spin-1/2 factorization. These relationships are summarized by the operator identities $(\square + m^2)\phi = 0$ for the scalar field and $(i\gamma^\mu \partial_\mu + m)(i\gamma^\nu \partial_\nu - m)\psi = (\square + m^2)\psi$ for the spinor.

Numerical and Computational Methods

Classical Integrators

The KGE's structure an exactly solvable linear part plus a mild nonlinearity—makes it ideal for exponential time-differencing and splitting schemes. Hochbruck and Ostermann's influential survey codified exponential integrators for evolution equations, demonstrating that Lawson and ETD schemes applied to semilinear wave-type problems achieve high-order accuracy with stability far superior to explicit Runge–Kutta methods under CFL constraints [8]. For highly oscillatory regimes where $\omega\Delta t \gg 1$, standard schemes must resolve every oscillation at prohibitive cost; Shi and Lubich introduced filtered finite difference methods for strongly oscillatory nonlinear KGEs that modulate the stiff linear flow analytically while treating the nonlinearity numerically, retaining accuracy over long times at cost independent of the oscillation frequency [45]. Spectral methods complement these advances in smooth periodic settings.

Long-time integration underpins the numerical study of oscillons. Gleiser and Sornborger's lattice simulations established the extreme longevity of three-dimensional oscillons in small periodic boxes, quantifying radiation rates across potential families [46], and Salmi and Hindmarsh characterized the radiation and relaxation channels of oscillons in detail [47]. Table 1 summarizes representative method families and their trade-offs.

Beyond raw accuracy, structure preservation distinguishes the successful schemes: splitting methods inherit the charge-conservation and symplectic character of the continuous flow, pseudo-spectral collocation exploits smoothness of periodic oscillon profiles for exponential convergence, and adaptive time stepping tracks the slow radiation tails without wasting effort on resolved interior oscillations [8]. The practical benchmark across all families is reproduction of the exponentially small oscillon decay rate over many periods a test that filters schemes sharply by their long-time error constants [45-47].

Domain	Role of the KGE	Key phenomena	Refs sources
Cosmology	Coherent scalar dynamics in expanding universe	Reheating, quintessence trackers, misalignment	[51] [52] [53] [54] [55] [56] [57]
Axion / scalar dark matter	Massive field in periodic potential	Misalignment, strings, axion stars	[7] [57] [58] [59]
Localized field configurations	Nonlinear self-interaction	Oscillons, breathers, radiation suppression	[6] [27] [28]
Condensed matter	Sine-Gordon collective modes	Soliton lattices, fluxons	[62] [63]
Dirac-material analogues	Massive relativistic dispersion	Klein tunneling, Hartman effect	[5] [64]
Strong-field gravity	Massive scalar on curved backgrounds	Superradiance, bound states	[17]

Table 1: Numerical and Computational Methods for the Klein–Gordon Equation

Machine Learning Approaches

Physics-informed neural networks embed the PDE residual directly into the training loss [9], and the KGE has become a standard benchmark within this framework because its second-order time structure stresses purely collocation-based losses. The same framework inverts naturally: unknown coefficients of the potential, initial conditions, or source terms can be recovered from sparse observations by extending the trainable parameters, an inverse capability that classical forward solvers do not provide out of the box [9]. Qian, Zhang, and Dong provided the first systematic error analysis and algorithmic remedies for PINNs applied to dynamic PDEs that are second order in time, identifying failure modes of naive formulations and proposing causal two-stage training with provable convergence rates [14]. Adaptive activation functions further accelerate convergence of both deep and physics-informed architectures [48]. Application-oriented studies confirm practical viability: Nouna and colleagues solved nonlinear KGE benchmark problems with the PyDEns neural PDE solver, reporting accuracy competitive with finite differences on smooth solutions, while Ferrer-Sánchez et al. extended PINN methodology to the gravitational collapse of a massless scalar field a characteristic-data problem in numerical relativity demonstrating that learned models can capture critical behavior near black-hole formation thresholds [15,49].

A canonical physics-informed neural network for the Klein–Gordon equation is a fully connected feed-forward network $\phi_\theta(\mathbf{x}, t)$ with several hidden layers and a smooth activation function, trained to minimize a composite loss $L = L_{\text{PDE}} + \lambda_{\text{ic}} L_{\text{ic}} + \lambda_{\text{bc}} L_{\text{bc}} + \lambda_{\text{data}} L_{\text{data}}$. The residual term L_{PDE} evaluates $\square\phi_\theta + V(\phi_\theta)$ at collocation points, with the second derivatives $\partial_t^2\phi_\theta$ and $\Delta\phi_\theta$ obtained by automatic differentiation rather than finite differences; the initial- and boundary-condition terms enforce $\phi_\theta(\mathbf{x}, 0) = \phi_0$ and $\partial_t\phi_\theta(\mathbf{x}, 0) = \phi_1$, and the optional data term assimilates observations. Because the second-order time derivative is the stiffest part of the residual, naive collocation training can converge to spurious solutions; Qian et al. [14] analyse this failure mode and propose a causal, two-stage training scheme with provable convergence rates, while adaptive activation functions [48] and

residual-based sampling improve conditioning. This architecture has become standard for the Klein–Gordon equation and its inverse problems [9,14,49].

Fractional Generalizations and Quantum Simulation

Fractional Klein–Gordon equations, combining spatial or temporal fractional derivatives with wave-type nonlocality, model anomalous diffusion in disordered media. Wang constructed new fractal soliton solutions for coupled fractional KGEs with beta-fractional derivatives via traveling-wave reductions, enlarging the known exact-solution catalogue [50]. On the quantum side, Jordan–Lee–Preskill algorithms compute scattering amplitudes in ϕ^4 theory with polynomial cost [10], and orbifold-lattice quantum simulation realizes gauge-scalar systems on near-term devices [26]. Together these directions indicate that the computational study of the KGE is now a three-way enterprise spanning classical analysis, machine learning, and quantum information.

Physical Applications

Cosmology: Inflation, Quintessence, Axions, and Oscillons

Scalar fields governed by the KGE are the primary dynamical actors in modern cosmology. Guth’s inflationary paradigm posits a slowly rolling scalar whose potential energy drives quasi-exponential expansion [51]; once inflation ends, the homogeneous field oscillates coherently about the minimum of its potential. Turner’s analysis of this regime showed that coherent scalar-field oscillations behave cosmologically as pressureless matter, $w = 0$, with amplitude diluting as $a^{-3/2}$ the kinematic foundation of reheating and cold dark matter from scalars. Ratra and Peebles introduced the inverse power-law potentials $V \propto \phi^{-\alpha}$ as quintessence candidates whose tracker solutions alleviate fine-tuning; subsequent bifurcation analyses mapped the full dynamical-systems phase structure of such Ratra–Peebles models and their extensions, and the definitive review of dark-energy dynamics codifies the role of Klein–Gordon fields across quintessence, kessence, and modified gravity [52–55].

Axion physics supplies the most compelling particle-physics realization. The invisible-axion solution to the strong CP problem produces a light pseudo-scalar whose early-universe misalignment is precisely coherent Klein–Gordon evolution in a cosine potential [56]. Marsh’s comprehensive review surveys axion cosmology from production mechanisms to observational signatures [7], while Klaer and Moore computed the dark-matter axion abundance including string-wall network contributions, sharpening mass bounds [57]. Simulations now track gravitational formation of relativistic axion stars from field fluctuations, and recent work derives oscillaton properties scalar configurations balancing gradient against periodic-potential forces from full Fourier expansions of exponential potentials [58–59]. On the laboratory side, axion-photon conversion in strongly magnetized plasmas, treated through coupled Klein–Gordon–Maxwell systems, underlies haloscope and helioscope detection strategies [16].

Oscillons occupy a special niche at the interface of analysis and cosmology. Amin et al. showed that generic single-field inflation models produce oscillons after inflation with model-independent phenomenology; Gleiser reviewed their applications across dimensions and inflationary settings; and Adib, Gleiser, and Almeida established existence and stability properties of asymmetric-bubble-derived oscillons [6,60,61]. Their longevity makes oscillons candidate seeds for primordial black holes and gravitational-wave sources.

An observational program is now consolidating around these scalar dynamics. Axion-star formation simulations connect primordial field fluctuations to present-day compact objects, oscillaton solutions provide equilibrium templates for scalar dark-matter halos, and axion-photon conversion calculations translate the underlying Klein–GordonMaxwell coupling into concrete sensitivity projections for haloscope experiments [16,58,59]. Superradiance constraints close the loop from the gravitational side: whenever a massive scalar exists, rotating black holes would spin down by populating bound states, so observed high-spin populations bound the allowed boson masses [17]. Each observable star mass function, halo profiles, laboratory conversion rates, spin distributions traces back to solutions of the Klein–Gordon equation in a different potential and background geometry.

Condensed Matter and Devices

Quasi-one-dimensional conductors realize sine–Gordon soliton lattices: Horowitz’s theory of polyacetylene and spin-Peierls systems predicted commensurate-incommensurate transitions governed by interacting soliton arrays, later confirmed by neutron and X-ray scattering [62]. Josephson junction chains support fluxon dynamics described by the same equation; Mazo and Ustinov review how engineered Josephson arrays implement sine–Gordon physics, including plasma resonances and fluxon propagation relevant to superconducting logic [63]. Breather excitations, though analytically exact only in the integrable limit, survive as long-lived resonances in weakly perturbed devices, providing a tunable platform on which the nonpersistence theorems can be probed experimentally through radiation damping measurements [42–43].

Graphene brings relativistic wave mechanics to the benchtop. Katsnelson, Novoselov, and Geim explained the anomalously transparent p-n interfaces of graphene as chiral tunneling the massless Dirac analogue of the Klein paradox with experiments confirming near-perfect transmission [5]. Bilayer graphene’s parabolic bands map instead onto massive relativistic kinematics; Park analyzed chiral tunneling and Hartman-effect tunneling times in bilayer structures, demonstrating Klein-paradox phenomenology in a tunable massive system whose dispersion mirrors the Klein–Gordon relation [64–67]. The historical paradox [18] thus re-emerged as an engineering resource: barrier transmission that classical mechanics forbids becomes a design principle for graphene-based electronics.

Gravitation and Black-Hole Physics

Massive scalar fields around rotating black holes trigger superradiant instabilities when the boson's Compton wavelength matches the horizon scale, populating the otherwise empty ergoregion with bound states. The mechanism is the bosonic analogue of stimulated emission: waves scattering off the rotating geometry can extract rotational energy when their frequency falls below the superradiant bound set by the horizon angular velocity. Yang and Miao computed superradiance spectra of massive scalar particles around rotating regular black holes, extending the classic Kerr analysis to nonsingular geometries and deriving exclusion plots analogous to those used to constrain ultralight boson masses [17]. Because the instability timescale is astronomically short for favorable mass couplings, absence of such spin-down across the observed pulsar and black-hole population translates into stringent exclusion regions—the most robust observational handle on very light Klein–Gordon fields [16–17]. At the level of fundamental unification, Kaluza–Klein compactifications generate tower upon tower of massive scalar modes whose dynamics is exactly Klein–Gordon, linking the equation to higher-dimensional gravity phenomenology [22]. Table 2 maps the principal application domains.

Domain	Role of the KGE	Key phenomena	Refs sources
Cosmology	Coherent scalar dynamics in expanding universe	Reheating, quintessence trackers, misalignment	[51] [52] [53] [54] [55] [56] [57]
Axion / scalar dark matter	Massive field in periodic potential	Misalignment, strings, axion stars	[7] [57] [58] [59]
Localized field configurations	Nonlinear self-interaction	Oscillons, breathers, radiation suppression	[6] [27] [28]
Condensed matter	Sine-Gordon collective modes	Soliton lattices, fluxons	[62] [63]
Dirac-material analogues	Massive relativistic dispersion	Klein tunneling, Hartman effect	[5] [64]
Strong-field gravity	Massive scalar on curved backgrounds	Superradiance, bound states	[17]

Table 2: Applications of the Klein–Gordon Equation (KGE) Across Physical Domains

OPEN PROBLEMS AND OUTLOOK

Three frontiers define the immediate agenda. First, rigorous analysis continues to expand its reach: the codimension-one stability program for focusing solitons and modified-scattering theory for quadratic nonlinearities in one dimension exemplify techniques that should extend to multi-dimensional and non-semilinear settings, while dispersive theory on cones, inverse-square backgrounds, and curved spacetimes more broadly remains incomplete away from flat space [11,12,29,30].

Second, computational methodology is converging on hybrid pipelines. The error analysis of PINNs for second-order-in-time hyperbolic problems now provides theoretical grounding for learned solvers, and demonstrations ranging from nonlinear benchmark equations to relativistic gravitational collapse suggest a division of labor in which classical integrators supply accuracy guarantees while neural surrogates handle inverse problems and uncertainty quantification [14,15,45,49]. Standardized benchmarks oscillon longevity, critical collapse, superradiant growth rates would accelerate cross-method comparison.

Third, quantum simulation of scalar field theories is approaching practical relevance. Polynomial-cost scattering algorithms and orbifold-lattice implementations anticipate hardware where quantum-corrected nonlinear dynamics including the possibly eternal quantum oscillons argued by Evslin can be probed directly rather than inferred from semiclassical estimates [10,26,28].

A fourth frontier is infrastructural. Reproducibility of learned solvers requires standardized datasets, shared error metrics, and reference classical solutions; the benchmark studies surveyed here point toward community-maintained test suites analogous to those that transformed numerical linear algebra [14,49]. Coupling differentiable solvers to automatic differentiation would let stability analyses and variational constructions be verified numerically at scale, closing the loop between rigorous theory and computation [11,32].

CONCLUSION

A century after its formulation, the Klein–Gordon equation functions simultaneously as historical artifact, mathematical laboratory, and working tool. Its 1926 genesis crystallized the marriage of quantum mechanics with special relativity; its interpretational difficulties forced the field-theoretic viewpoint that defines modern particle physics; its linear theory anchors dispersive analysis, and its nonlinear variants sustain one of the deepest programs in PDE theory from Klainerman's small-data global existence to contemporary stability manifolds. Numerically, it benchmarks exponential integrators, filtered schemes, physics-informed networks, and quantum algorithms alike. Physically, it governs coherent fields across twenty decades of scale, from reheating oscillons to Josephson fluxons and black-hole bound states, and each of these arenas is now observationally or experimentally accessible. The equation's trajectory suggests that another century of Klein–Gordon research will be driven less by the equation itself than by the new mathematics, computation, and experiment it continues to catalyze.

REFERENCES

1. Klein O. Quantentheorie und fünfdimensionale Relativitätstheorie. *Z Phys.* 1926;37(12):895-906.
2. Gordon W. Der Comptoneffekt nach der Schrödingerschen Theorie. *Z Phys.* 1926;40(1-2):117-133.
3. de Broglie L. Recherches sur la théorie des quanta. *Ann Phys (Paris).* 1925;10(3):22-128.
4. Peskin ME, Schroeder DV. An introduction to quantum field theory. Reading (MA): Addison-Wesley; 1995.
5. Katsnelson MI, Novoselov KS, Geim AK. Chiral tunnelling and the Klein paradox in graphene. *Nat Phys.* 2006;2(9):620-625.
6. Amin MA, Easther R, Finkel H, Flauger R, Hertzberg MP. Oscillons after inflation. *Phys Rev Lett.* 2012;108(24):241302.
7. Marsh DJE. Axion cosmology. *Phys Rep.* 2016;643:1-79.
8. Hochbruck M, Ostermann A. Exponential integrators. *Acta Numer.* 2010;19:209-286.
9. Raissi M, Perdikaris P, Karniadakis GE. Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations. *J Comput Phys.* 2019;378:686-707.
10. Jordan SP, Lee KSM, Preskill J. Quantum algorithms for quantum field theories. *Science.* 2012;336(6085):1130-1133.
11. Lührmann J, Schlag W. On codimension one stability of the soliton for the 1D focusing cubic Klein-Gordon equation. *Commun Am Math Soc.* 2024;4(7):230-356.
12. Germain P, Pusateri F. Quadratic Klein-Gordon equations with a potential in one dimension. *Forum Math Pi.* 2022;10:e9.
13. Beceanu M, Chen G. Strichartz estimates for the Klein-Gordon equation in three and higher dimensions. *Pure Appl Anal.* 2022;4(4):767-809.
14. Qian Y, Zhang Y, Huang Y, Dong S. Physics-informed neural networks for approximating dynamic (hyperbolic) PDEs of second order in time: Error analysis and algorithms. *J Comput Phys.* 2023;495:112527.
15. Ferrer-Sánchez A, Villanueva-Espinosa N, Hernani-Morales C, Ruiz de Austri-Bazan R, Font JA, Martin-Guerrero JD, et al. Addressing the gravitational collapse of a massless scalar field with physics-informed neural networks. *Mach Learn Sci Technol.* 2026;7(2):025038.
16. Millar AJ, Baum S, Lawson M, Marsh MCD. Axion-photon conversion in strongly magnetised plasmas. *J Cosmol Astropart Phys.* 2021;2021(11):013.
17. Yang H, Miao YG. Superradiance of massive scalar particles around rotating regular black holes. *Chin Phys C.* 2023;47(7):075101.
18. Klein O. Die Reflexion von Elektronen an einem Potentialsprung nach der relativistischen Dynamik von Dirac. *Z Phys.* 1929;53(3-4):157-165.
19. Dirac PAM. The quantum theory of the electron. *Proc R Soc Lond A.* 1928;117(778):610-624.
20. Bjorken JD, Drell SD. Relativistic quantum mechanics. New York: McGraw-Hill; 1964.
21. Feshbach H, Villars F. Elementary relativistic wave mechanics of spin 0 and spin 1/2 particles. *Rev Mod Phys.* 1958;30(1):24-45.
22. Overduin JM, Wesson PS. Kaluza-Klein gravity. *Phys Rep.* 1997;283(5-6):303-378.
23. Englert F, Brout R. Broken symmetry and the mass of gauge vector mesons. *Phys Rev Lett.* 1964;13(9):321-323.
24. Higgs PW. Broken symmetries and the masses of gauge bosons. *Phys Rev Lett.* 1964;13(16):508-509.
25. Guralnik GS, Hagen CR, Kibble TWB. Global conservation laws and massless particles. *Phys Rev Lett.* 1964;13(20):585-587.
26. Buser AJ, Gharibyan H, Hanada M, Honda M, Liu J. Quantum simulation of gauge theory via orbifold lattice. *J High Energy Phys.* 2021;2021(9):034.
27. Hertzberg MP. Quantum radiation of oscillons. *Phys Rev D.* 2010;82(4):045022.
28. Evslin J, Romańczukiewicz T, Wereszczyński A. Quantum oscillons may be long-lived. *J High Energy Phys.* 2023;2023(8):182.
29. Lee H, Seo I, Seok J. Local smoothing and Strichartz estimates for the Klein-Gordon equation with the inverse-square potential. *Discrete Contin Dyn Syst.* 2020;40(1):597-608.
30. Yin Z, Zhang F. Decay and Strichartz estimates for Klein-Gordon equation on a cone I: Spinless case. *Forum Math.* 2024;36(3):811-823.
31. Nakanishi K, Schlag W. Invariant manifolds and dispersive Hamiltonian evolution equations. Zürich: European Mathematical Society; 2011.
32. Shatah J, Struwe M. Geometric wave equations. Providence (RI): American Mathematical Society; 1998. Courant Lecture Notes in Mathematics; vol. 2.
33. Miao C, Zheng J. Energy scattering for a Klein-Gordon equation with a cubic convolution. *J Differential Equations.* 2014;257(7):2322-2362.
34. Derrick GH. Comments on nonlinear wave equations as models for elementary particles. *J Math Phys.* 1964;5(9):1252-1254.
35. Strauss WA. Existence of solitary waves in higher dimensions. *Commun Math Phys.* 1977;55(2):149-162.
36. Shatah J. Stable standing waves of nonlinear Klein-Gordon equations. *Commun Math Phys.* 1983;91(3):313-327.
37. Shatah J. Unstable ground state of nonlinear Klein-Gordon equations. *Trans Am Math Soc.* 1985;290(2):701-710.
38. Shatah J. Normal forms and quadratic nonlinear Klein-Gordon equations. *Commun Pure Appl Math.* 1985;38(5):685-696.
39. Klainerman S. Global existence of small amplitude solutions to nonlinear Klein-Gordon equations in four space-time dimensions. *Commun Pure Appl Math.* 1985;38(5):631-641.
40. Ginibre J, Velo G. The global Cauchy problem for the non linear Klein-Gordon equation. *Math Z.* 1985;189(4):487-505.
41. Ginibre J, Velo G. The global Cauchy problem for the non linear Klein-Gordon equation—II. *Ann Inst H Poincaré C Anal Non Linéaire.* 1989;6(1):15-35.
42. Segur H, Kruskal MD. Nonexistence of small-amplitude breather solutions in ϕ^4 theory. *Phys Rev Lett.* 1987;58(8):747-750.
43. Denzler J. Nonpersistence of breather families for the perturbed sine Gordon equation. *Commun Math Phys.* 1993;158(2):397-430.
44. Johnson S, Biswas A. Breather dynamics of the sine-Gordon equation. *Communications in Theoretical Physics.* 2013;59(6):664-670.

45. Shi Y, Lubich C. A filtered finite difference method for a highly oscillatory nonlinear Klein–Gordon equation. arXiv Preprint. 2025. arXiv: 2504.19359. DOI
46. Gleiser M, Sornborger A. Long-lived localized field configurations in small lattices: Application to oscillons. *Phys Rev E*. 2000;62(1):1368-1374.
47. Salmi P, Hindmarsh M. Radiation and relaxation of oscillons. *Phys Rev D*. 2012;85(8):085033.
48. Jagtap AD, Kawaguchi K, Karniadakis GE. Adaptive activation functions accelerate convergence in deep and physics-informed neural networks. *J Comput Phys*. 2020;404:109136.
49. Nouna S, Tammouch I, Nouna A, Mansouri M. Solving nonlinear Klein–Gordon equations via PyDEns: A neural network-based PDE solver. *Informatica*. 2025;49(28).
50. Wang K. New fractal soliton solutions for the coupled fractional Klein–Gordon equation with β -fractional derivative. *Fractals*. 2023;31(1):2350003.
51. Guth AH. Inflationary universe: A possible solution to the horizon and flatness problems. *Phys Rev D*. 1981;23(2):347-356.
52. Turner MS. Coherent scalar-field oscillations in an expanding universe. *Phys Rev D*. 1983;28(6):1243-1247.
53. Ratra B, Peebles PJE. Cosmological consequences of a rolling homogeneous scalar field. *Phys Rev D*. 1988;37(12):3406-3427.
54. Humieja F, Szydlowski M. Bifurcations in Ratra–Peebles quintessence models and their extensions. *Eur Phys J C*. 2019;79(9):794.
55. Copeland EJ, Sami M, Tsujikawa S. Dynamics of dark energy. *Int J Mod Phys D*. 2006;15(11):1753-1935.
56. Preskill J, Wise MB, Wilczek F. Cosmology of the invisible axion. *Phys Lett B*. 1983;120(1–3):127-132.
57. Klaer VB, Moore GD. The dark-matter axion mass. *J Cosmol Astropart Phys*. 2017;2017(11):049.
58. Widdicombe JY, Helfer T, Marsh DJE, Lim EA. Formation of relativistic axion stars. *J Cosmol Astropart Phys*. 2018;2018(10):005.
59. Mahmoodzadeh A, Ghaderi K, Amiri P. Oscillatons in scalar-field dark matter from a full Fourier expansion of an exponential potential. *Phys Dark Universe*. 2026;52:102314.
60. Gleiser M. Oscillons in scalar field theories: Applications in higher dimensions and inflation. *Int J Mod Phys D*. 2007;16(02n03):219-229.
61. Adib AB, Gleiser M, Almeida CAS. Long-lived oscillons from asymmetric bubbles: Existence and stability. *Phys Rev D*. 2002;66(8):085011.
62. Horowitz B. Soliton lattice in polyacetylene, spin-Peierls systems, and two-dimensional sine-Gordon systems. *Phys Rev Lett*. 1981;46(11):742-745.
63. Mazo JJ, Ustinov AV. The sine-Gordon equation in Josephson-junction arrays. Cham: Springer; 2014.
64. Park CS. Chiral tunneling, tunneling times, and Hartman effect in bilayer graphene. *Phys Rev B*. 2014;89(11):115423.
65. Aftab MN, Butt SI, Seol Y. Fundamentals of dual basic symmetric quantum calculus and its fractional perspectives. *Fractal Fract*. 2025;9(4):237.
66. Aftab MN, Abu-Zinadah H, Lupas AA, Lee CC. New (p, q) -symmetric Hermite–Hadamard-type inequalities in quantum calculus. *Int J Appl Math Optim Artif Intell*. 2026;1(01):1-10.
67. Aftab MN, Butt SI, Alammam M, Seol Y. Analysis of quantum multiplicative calculus and related inequalities. *Mathematics*. 2025;13(21):3381.