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Application of the J -Transform for Solving Volterra Integral Equations of the First and Second Kind

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ABSTRACT

This paper presents a new use of the J -transform to solve both first and second kind Volterra integral equations. Several examples are provided to support the methodological applicability and efficiency. The empirical evidence shows that the transform of the form of a J - is a powerful and effective tool of solving Volterra integral equations in applied mathematics and also in the field of engineering.

Keywords: Volterra Integral Equations; J -Transform; Convolution Theorem; Integral Transforms; Operational Calculus

INTRODUCTION

The modelling of systems that depend upon previous events is a natural model for applied mathematics and hence integral equations have a significant role in applied mathematics. Memory-dependent behaviour occurs in a variety of physical, biological and engineering systems. In these situations, sometimes, differential equations are not adequate; at other times, an integral formulation is adequate to take into account accumulated effects of previous states. Vito Volterra introduced a class of integral equations known as Volterra integral equations (VIEs), which play a particularly important role among the various classes of integral equations. They can be used to describe processes in which the state of a system at time (x) is dependent upon its history in the interval [0,x] and are thus well suited to the modelling of phenomena with memory [1, 2].

Volterra integral equations are basically categorised as first kind and second kind equations based on the position of the unknown function. In first-kind equations the unknown is present only in the integral, in second-kind equations the unknown appears both inside and outside the integral. This distinction is not obvious but it has significance in mathematics. In general, first kind equations are ill-posed in the sense of Hadamard, and this means that they could be subject to large variation in the solution when the input data are varied slightly. Second-kind equations with well-behaved enough kernels, on the other hand, are typically well-posed and have unique and stable solutions under fairly mild conditions. Therefore, the analytical and numerical solutions to the equation are affected by the form of the equation itself [1].

Many years have passed and the mathematical theory of Volterra integral equations has grown significantly, much attention has been paid to the existence, uniqueness, and stability of solutions. The classical results are mostly relying on the Banach contraction principle for the Volterra operator in a suitable function space. A property of this operator is that it is quasi-nilpotent, that is, repeated application can always produce a contraction even if the kernel is not small Lipschitz. It has become a bedrock of the classical theory, and it plays an important role in many current developments.

Based on this, the theory has been extended in a number of directions. General-ized Banach-type fixed-point theorems have been employed to obtain the existence and uniqueness results of nonlinear Volterra–Fredholm equations without imposing any global contraction condition [3]. The same results can be achieved, with fuzzy metric fixed-point theory, in nonlinear Volterra and Fredholm equations [4]. For timescales, there are studies that use the Banach fixed point theorem and a Bielecki type norm and more recent studies with the sum of operators to extend these results [5-6]. Other contributions are related to the resolvent based existence theory of Volterra equations on time scales and the use of the Arzel–Ascoli theorem, Schauder's fixed-point theorem and Gronwall-type inequalities in order to obtain existence results for nonlinear convolution-type integral equations. The studies have added to the theoretical base of the Volterra integral equations and expanded their use.

In parallel to these theoretical advances, a number of solution techniques have been the target of much research. Since convolution operations are transformed into simple algebraic products, the use of integral transforms continues to be one of the best methods to solve linear Volterra integral equations involving convolution kernels. This conversion makes solving the problems much easier. The classical example is the Laplace transform and it is still one of the most commonly used methods in this field [7].

Many Laplace-type transforms have been developed over the past 30 years to retain the convolution property, while adding flexibility and/or computational efficiency. The instances of well-known transforms are the Sumudu transform , the Elzaki transform and the natural transform [4, 6, 8]. The Rishi transform , the Bayawa transform and a few unified transform frameworks that comprise the Laplace, Sumudu, and Elzaki transforms as special cases, have additionally increased the space of this study [9-11].

The following transform is applied to a number of Volterra-type equations and has proven successful. The Sumudu transform and Adomian decomposition method has been employed to solve systems of nonlinear Volterra integral equations as an example [8]. These have also been used for coupled systems of integral and ordinary differ-ential equations by Sumudu and Elzaki [12]. In the same fashion, the Rishi transform has been used in solving nonlinear first-kind Volterra integral equations and re-cently mixed Volterra–Fredholm integro-differential equations [11, 13]. The transform also has been found to be suitable for the second kind linear Volterra integro-differential equa-tions (Volterra equations) [14]. In addition to these analytical methods, there have also been further developments in the field of numerical and semi-analytical methods. Haar-wavelet collocation methods have been found to be very effective in solving Volterra and Volterra–Fredholm equations, such as Volterra–Fredholm equations with fractional integro-differential models and delay Volterra–Fredholm equations [15-16]. The develop-ments taken together show the continued relevance of Volterra integral equations and the search for suitable solutions with mathematically valid and at the same time numerically efficient solution methods.

In this context, the J -transform recently developed by Shehu and Zhao is one of the latest members of the family of operational integral transforms. The transform has advantages of certain earlier transform algorithms and provides operationally practical qualities such as linearity, scaling, differentiation and convolution thereby providing a direct and efficient analytical solution procedure [17]. General solution formulas are derived for both first-kind and second-kind Volterra integral equations, and several illustrative examples are presented to demonstrate the effectiveness and computational simplicity of the proposed technique. The principal contributions of this paper are fourfold [18-19]. First, the operational framework of the J -transform is extended to the analytical solution of Volterra integral equations. Second, general transform-domain formulations are derived for both first-kind and second-kind convolution Volterra equations. Third, the appli-cability of the proposed methodology is validated through representative examples that recover exact analytical solutions. Finally, the present work broadens the range of appli-cations of the J -transform and provides a foundation for future investigations involving nonlinear, fractional, Volterra–Fredholm, and integro-differential equations [20-22].

Properties of the J -Transform

Definition

Let $f : [0, \infty) \to \mathbb{R}$ be of exponential order. The $\mathcal{J}$-transform of $f(t)$ is defined by:

$$\mathcal{J}\{f(t)\} = F(s,u) = u\int_0 \quad e^{-st/u} f(t)\,dt, \quad s > 0, u > 0. \tag{1}$$

Basic Properties

Linearity property. Let f and g be in set A. It holds that

$$\mathcal{J}\{af(t) + bg(t)\} = a\,\mathcal{J}\{f(t)\} + b\,\mathcal{J}\{g(t)\},$$

where a and b are constants.

Proof.

Let

$$\mathcal{J}\{f(t)\} = u\int_0^\infty e^{-st/u} f(t)\,dt.$$

Consider

$$\mathcal{J}\{af(t)+bg(t)\} = u\int_0^\infty e^{-st/u}[af(t)+bg(t)]\,dt.$$

Using the linearity of integration,

$$\mathcal{J}\{af(t)+bg(t)\} = u\left[a\int_0^\infty e^{-st/u}f(t)\,dt + b\int_0^\infty e^{-st/u}g(t)\,dt\right].$$

Taking the constants outside the integrals, we get

$$\mathcal{J}\{af(t)+bg(t)\} = a\left(u\int_0^\infty e^{-st/u}f(t)\,dt\right) + b\left(u\int_0^\infty e^{-st/u}g(t)\,dt\right).$$

Since

$$\mathcal{J}\{f(t)\} = F(s,u), \qquad \mathcal{J}\{g(t)\} = G(s,u),$$

we obtain

$$\mathcal{J}\{af(t)+bg(t)\} = aF(s,u) + bG(s,u).$$

Hence,

$$\mathcal{J}\{af(t)+bg(t)\} = a\mathcal{J}\{f(t)\} + b\Im\{g(t)\}.$$

Therefore, the theorem is proved.

First translation or shifting property of J-transform: Let $\exp(\alpha t)v(t) \in A$, where α is constant. Then

$$J\left[\exp(\alpha t)v(t)\right] = \frac{s-\alpha u}{s}V\left(s, \frac{su}{s-\alpha u}\right).$$

Proof: By Definition , we have

$$J\left[\exp(\alpha t)v(t)\right] = u^2\int_0^\infty \exp\big(-(s-\alpha u)t\big)v(ut)\,dt.$$

Let

$$\eta s = (s-\alpha u)t,$$

which implies

$$t = \frac{s\eta}{s-\alpha u}, \qquad dt = \frac{s}{s-\alpha u}\, d\eta.$$

$$\begin{aligned} J\left[\exp(\alpha t)v(t)\right] &= \frac{su^2}{s-\alpha u}\int_0^\infty \exp(-\eta s)v\left(\frac{us\eta}{s-\alpha u}\right) d\eta \\ &= \frac{su^2}{s-\alpha u}\int_0^\infty \exp(-st)v\left(\frac{ust}{s-\alpha u}\right) dt \\ &= \frac{u^2 s}{s-\alpha u}\left(\frac{s-\alpha u}{us}\right)^2 V\left(s, \frac{su}{s-\alpha u}\right) \\ &= \frac{s-\alpha u}{s} V\left(s, \frac{su}{s-\alpha u}\right). \end{aligned}$$

This ends the proof.

2. **Scaling property:** Let $F = F(s,u)$ be the $\mathcal{J}$-transform of the function $f = f(t)$, and $a > 0$. Then we have the scaling property

$$\mathcal{J}\{f(at)\} = \frac{1}{a} f\left(\frac{s}{a}, u\right).$$

Property 3.3 (Scaling property): Let $V = V(s,u)$ be the J-transform of the function $v = v(t)$, and $\alpha > 0$. Then we have the scaling property

$$J[v(\alpha t)] = \frac{1}{\alpha} V\left(\frac{s}{\alpha}, u\right).5$$

$$J[v(\alpha t)] = u\int_0^\infty \exp\left(-\frac{st}{u}\right) v(\alpha t)\, dt.$$

Substituting $\eta = \alpha t$, which implies

$$t = \frac{\eta}{\alpha} \qquad \text{and} \qquad dt = \frac{d\eta}{\alpha},$$

$$\begin{aligned} J[v(\alpha t)] &= \frac{u}{\alpha}\int_0^\infty \exp\left(-\frac{s\eta}{u\alpha}\right) v(\eta)\, d\eta \\ &= \frac{u}{\alpha}\int_0^\infty \exp\left(-\frac{st}{u\alpha}\right) v(t)\, dt \\ &= \frac{u}{\alpha}\int_0^\infty \exp\left(-\frac{\left(\frac{s}{\alpha}\right)t}{u}\right) v(t)\, dt \\ &= \frac{1}{\alpha} V\left(\frac{s}{\alpha}, u\right). \end{aligned}$$

The proof ends.

3. **Inverse property of $\mathcal{J}$-transform:**
   Let $F(s,u)$ be the $\mathcal{J}$-transform of the function $f(t)$. $\mathcal{J}^{-1}$ is called the inverse $\mathcal{J}$-transform of $F(s,u)$, that is,

$$\mathcal{J}^{-1}\{F(s,u)\} = f(t).$$

$$\mathcal{J}^{-1}\{f(s,u)\} = F(t)$$

Theorem

Assume that $f^{(i)} \in A$, $i = 0, 1, 2, \ldots, n$. Let $F(s,u)$ and $F_n(s,u)$ be the transforms of f and $f^{(n)}$, respectively. Then

$$F_n(s,u) = \mathcal{J}\{f^{(n)}(t)\} = \frac{s^n}{u^n}F(s,u) - \sum_{k=0}^{n-1} \frac{s^{\,n-(k+1)}}{u^{\,n-k}}\, f^{(k)}(0).$$

Proof of Theorem 2.3

Let

$$F(s,u) = \mathcal{J}\{f(t)\} = u\int_0^\infty e^{-st/u} f(t)\, dt,$$

and let

$$F_n(s,u) = \mathcal{J}\{f^{(n)}(t)\}.$$

We prove the theorem by mathematical induction.

For $n = 1$:
By definition,

$$F_1(s,u) = u\int_0^\infty e^{-st/u} f'(t)\, dt.$$

Integrating by parts, let

$$v = e^{-st/u}, \qquad dw = f'(t)\, dt.$$

Then,

$$dv = -\frac{s}{u}e^{-st/u}\, dt, \qquad w = f(t).$$

Therefore,

$$\begin{aligned}\int_0^\infty e^{-st/u} f'(t)\, dt &= \left[e^{-st/u} f(t)\right]_0^\infty - \int_0^\infty f(t)\left(-\frac{s}{u}e^{-st/u}\right) dt \\ &= -f(0) + \frac{s}{u}\int_0^\infty e^{-st/u} f(t)\, dt.\end{aligned}$$

Multiplying by u, we obtain

$$F_1(s,u) = -uf(0) + s\int_0^{\infty} e^{-st/u} f(t)\,dt$$
$$= -uf(0) + \frac{s}{u}F(s,u).$$

Hence,

$$F_1(s,u) = \frac{s}{u}F(s,u) - uf(0).$$

Thus, the result holds for $n = 1$.

For $n = 2$:
Applying the previous result to $f'(t)$, we get

$$F_2(s,u) = \frac{s}{u}F_1(s,u) - uf'(0).$$

Substituting the value of $F_1(s,u)$,

$$F_2(s,u) = \frac{s}{u}\left(\frac{s}{u}F(s,u) - uf(0)\right) - uf'(0)$$
$$= \frac{s^2}{u^2}F(s,u) - sf(0) - uf'(0).$$

Induction Hypothesis:
Assume that for some positive integer n,

$$F_n(s,u) = \frac{s^n}{u^n}F(s,u) - \sum_{k=0}^{n-1} \frac{s^{n-k-1}}{u^{n-k-2}} f^{(k)}(0).$$

Induction Step: We shall prove that

$$F_{n+1}(s,u) = \frac{s^{n+1}}{u^{n+1}}F(s,u) - \sum_{k=0}^{n} \frac{s^{n-k}}{u^{n-k-1}} f^{(k)}(0).$$

Using the first derivative formula,

$$F_{n+1}(s,u) = \frac{s}{u}F_n(s,u) - uf^{(n)}(0).$$

Substituting the induction hypothesis,

$$F_{n+1}(s,u) = \frac{s}{u}\left[\frac{s^n}{u^n}F(s,u) - \sum_{k=0}^{n-1} \frac{s^{n-k-1}}{u^{n-k-2}} f^{(k)}(0)\right] - uf^{(n)}(0)$$
$$= \frac{s^{n+1}}{u^{n+1}}F(s,u) - \sum_{k=0}^{n-1} \frac{s^{n-k}}{u^{n-k-1}} f^{(k)}(0) - uf^{(n)}(0).$$

Combining the last term into the summation gives

$$F_{n+1}(s,u) = \frac{s^{n+1}}{u^{n+1}}F(s,u) - \sum_{k=0}^{n} \frac{s^{n-k}}{u^{n-k-1}} f^{(k)}(0).$$

Hence,

$$F_n(s,u) = \frac{s^n}{u^n}F(s,u) - \sum_{k=0}^{n-1} \frac{s^{n-k-1}}{u^{n-k-2}} f^{(k)}(0).$$

Therefore, the theorem is proved.

$$\boxed{F_n(s,u) = \frac{s^n}{u^n}F(s,u) - \sum_{k=0}^{n-1} \frac{s^{n-k-1}}{u^{n-k-2}} f^{(k)}(0)}$$

Convolution Theorem

Let the functions $F(t)$ and $G(t)$ belong to A. If $F(s,u)$ and $G(s,u)$ are the respective $\mathcal{J}$-transforms of $F(t)$ and $G(t)$, then the convolution theorem of the $\mathcal{J}$-transform is given by

$$\mathcal{J}\{(f*g)(t)\} = \frac{1}{u}\, f(s,u)\, g(s,u).$$

Where $f*g$ is the convolution of two functions $f(t)$, which is defined by

$$(f*g)(x) = \int_0^x f(t)\, g(x-t)\, dt. \tag{2}$$

Proof:

By the definition of the $\mathcal{J}$-transform,

$$\mathcal{J}\{(f*g)(x)\} = u\int_0^\infty e^{-sx/u}(f*g)(x)\, dx.$$

Substituting

$$(f*g)(x) = \int_0^x f(t)g(x-t)\, dt,$$

we obtain

$$\mathcal{J}\{(f*g)(x)\} = u\int_0^\infty e^{-sx/u}\left(\int_0^x f(t)g(x-t)\, dt\right) dx.$$

Interchanging the order of integration,

$$\mathcal{J}\{(f*g)(x)\} = u\int_0^\infty \int_t^\infty e^{-sx/u} f(t)g(x-t)\, dx\, dt.$$

Let

$$y = x - t.$$

Then

$$x = t + y, \qquad dx = dy.$$

Hence,

$$\mathcal{J}\{(f*g)(x)\} = u\int_0^\infty \int_0^\infty e^{-s(t+y)/u} f(t)g(y)\, dy\, dt.$$

Therefore,

$$\mathcal{J}\{(f*g)(x)\} = u\left(\int_0^\infty e^{-st/u} f(t)\,dt\right)\left(\int_0^\infty e^{-sy/u} g(y)\,dy\right).$$

Since

$$F(s,u) = u\int_0^\infty e^{-st/u} f(t)\,dt,$$

and

$$G(s,u) = u\int_0^\infty e^{-sy/u} g(y)\,dy,$$

we obtain

$$\mathcal{J}\{(f*g)(x)\} = \frac{1}{u}F(s,u)G(s,u).$$

Hence,

$$\boxed{\mathcal{J}\{(f*g)(x)\} = \frac{1}{u}F(s,u)G(s,u)}.$$

Table of Common J-Transforms

S/N	Function	$\mathcal{J}$-Transform
1	1	$\frac{u^2}{s}$
2	t	$\frac{u^3}{s^2}$
3	t^n	$\frac{n!\,u^{n+2}}{s^{n+1}}$
4	e^{at}	$\frac{u^2}{s-au}$
5	$\sin at$	$\frac{au^3}{s^2+a^2u^2}$
6	$\cos at$	$\frac{su^3}{s^2+a^2u^2}$
7	$\sinh at$	$\frac{au^3}{s^2-a^2u^2}$
8	$\cosh at$	$\frac{su^3}{s^2-a^2u^2}$
9	$\frac{\sin at}{a}$	$\frac{u^3}{s^2+a^2u^2}$
10	$e^{bt}\cosh(at)$	$\frac{u^2(s-bu)}{(s-bu)^2-a^2u^2}$
11	$e^{bt}\sinh(at)$	$\frac{u^3}{(s-bu)^2-a^2u^2}$
12	$t\sinh(at)$	$\frac{2au^4s}{(s^2-a^2u^2)^2}$
13	$\frac{e^{bt}\sinh(at)}{a}$	$\frac{u^3}{(s-bu)^2-a^2u^2}$
14	$\frac{t\sin(at)}{2a}$	$\frac{u^4s}{(s^2+a^2u^2)^2}$

Derivation of Volterra Integral Equations Using $\mathcal{J}$- Transform

First-Kind VIE

$$f(x) = \beta \int_0^x k(x,t)\, u(t)\, dt. \tag{1}$$

Applying $\mathcal{J}$-Transform to equation (1), we have

$$\mathcal{J}\{f(x)\} = \mathcal{J}\left\{\beta \int_0^x k(x,t)\, u(t)\, dt\right\}.$$

$$\Rightarrow \quad f(s,u) = \frac{\beta\, k(s,u)\, V(s,u)}{u}.$$

$$\Rightarrow \quad uf(s,u) = \beta\, k(s,u)\, V(s,u).$$

$$\Rightarrow \quad V(s,u) = \frac{uf(s,u)}{\beta\, k(s,u)}.$$

Taking inverse of $\mathcal{J}$-transform, we get

$$\mathcal{J}^{-1}\{V(s,u)\} = \mathcal{J}^{-1}\left\{\frac{uf(s,u)}{\beta\, k(s,u)}\right\}.$$

$$U(t) = \mathcal{J}^{-1}\left\{\frac{uf(s,u)}{\beta\, k(s,u)}\right\}.$$

Second-Kind VIE

$$U(x) = g(x) + \beta \int_0^x k(x,t) U(t)\, dt$$

Applying $\mathcal{J}$-transform and using convolution theorem:

$$V(s,u) = \mathcal{J}\{U(x)\} = \mathcal{J}\{g(x)\} + \frac{\beta}{u} k(s,u) V(s,u)$$

$$\Rightarrow V(s,u)\left(1 - \frac{\beta}{u} k(s,u)\right) = \mathcal{J}\{g(x)\}$$

$$\Rightarrow V(s,u) = \frac{\mathcal{J}\{g(x)\}}{1 - \frac{\beta}{u} k(s,u)}$$

Inverse transform gives the solution:

$$U(x) = \mathcal{J}^{-1}\left\{\frac{\mathcal{J}\{g(x)\}}{1 - \frac{\beta}{u} k(s,u)}\right\}.$$

Examples of Volterra Integral Equations

Introduction

In this chapter some examples will be presented to explain the derivation of $\mathfrak{J}$–transform in solving linear Volterra integral and integrodifferential equations of first and second kind.

Example 1:

Solve the Volterra integral equation of first kind using $\mathfrak{J}$–transform:

$$1 + x - e^x = \int_0^x (t - x)u(t)\, dt$$

Taking $\mathfrak{J}$–transform,

$$\mathfrak{J}\{1 + x - e^x\} = \mathfrak{J}\left\{\int_0^x (t - x)u(t)\, dt\right\}$$

$$\Rightarrow \mathfrak{J}\{1\} + \mathfrak{J}\{x\} - \mathfrak{J}\{e^x\} = \frac{1}{u}\mathfrak{J}\{-x\}\mathfrak{J}\{u(t)\}$$

$$\Rightarrow \frac{u^2}{s} + \frac{u^3}{s^2} + \frac{u^2}{s-u} = \frac{1}{u}\left(-\frac{u^3}{s^2}\right)V(s,u)$$

$$\Rightarrow V(s,u) = \frac{u^2}{s-u}$$

Taking inverse $\mathfrak{J}$ transform,

$$U(x) = e^x$$

Example 2: Using $\mathfrak{J}$–Transform

$$1 - \cos x = \int_0^x \cos(x - t)u(t)\, dt$$

Taking $\mathfrak{J}$ transform to both sides,

$$\mathfrak{J}\{1 - \cos x\} = \mathfrak{J}\left\{\int_0^x \cos(x - t)u(t)\, dt\right\}$$

$$\Rightarrow \frac{u^2}{s} - \frac{u^2 s}{s^2 + u^2} = \frac{1}{u}\left(\frac{u^2 s}{s^2 + u^2}\right)V(s,u)$$

$$\Rightarrow V(s,u) = \frac{u^4}{us^2}$$

Taking inverse transform,

$$U(x) = x$$

Example 3:

$$\sinh x = \int_0^x e^{(x-t)} u(t)\, dt$$

Taking $\mathfrak{J}$ transform,

$$\mathfrak{J}\{\sinh x\} = \mathfrak{J}\left\{\int_0^x e^{x-t} u(t)\, dt\right\}$$

$$\Rightarrow \frac{u^3}{s^2-u^2} = \frac{1}{u}\mathfrak{J}\{e^x\}\mathfrak{J}\{u(t)\}$$

$$\Rightarrow V(s,u) = \frac{u^2}{s+u}$$

Thus,

$$U(x) = e^x$$

Example 4: Solve the Volterra integral equation of the first kind

$$\int_0^t (t-\tau)e^{t-\tau} x(\tau)\, d\tau = t^2 e^t.$$

Solution:

Observe that the kernel of the integral equation is

$$k(t) = te^t.$$

Hence, the equation can be expressed in convolution form as

$$(k * x)(t) = t^2 e^t.$$

Applying the $\mathcal{J}$-transform to both sides yields

$$\mathcal{J}\{k * x\} = \mathcal{J}\{t^2 e^t\}.$$

Using the convolution theorem, we have

$$\frac{1}{u} K(s,u) V(s,u) = \mathcal{J}\{t^2 e^t\},$$

where

$$K(s,u) = \mathcal{J}\{te^t\}$$

and

$$V(s,u) = \mathcal{J}\{x(t)\}.$$

To determine $K(s,u)$, recall that

$$\mathcal{J}\{t\} = \frac{u^3}{s^2}.$$

Applying the shifting property gives

$$K(s,u) = \mathcal{J}\{te^t\} = \frac{s-u}{s} F\left(s, \frac{su}{s-u}\right),$$

where

$$F(s,u) = \frac{u^3}{s^2}.$$

Substituting $F(s,u)$ into the above expression, we get

$$K(s,u) = \frac{s-u}{s}\left[\frac{\left(\frac{su}{s-u}\right)^3}{s^2}\right] = \frac{u^3 s}{(s-u)^2}.$$

Next, since

$$\mathcal{J}\{t^2\} = \frac{2u^4}{s^3},$$

the transform of the right-hand side becomes

$$\mathcal{J}\{t^2e^t\} = \frac{s-u}{s}\left[\frac{2\left(\frac{su}{s-u}\right)^4}{s^3}\right] = \frac{2u^4 s}{(s-u)^3}.$$

Substituting these results into the transformed equation gives

$$\frac{1}{u}\left(\frac{u^3 s}{(s-u)^2}\right) V(s,u) = \frac{2u^4 s}{(s-u)^3}.$$

Solving for $V(s,u)$, we find that

$$V(s,u) = \frac{2u^2}{s-u}.$$

Taking the inverse $\mathcal{J}$-transform of both sides, we obtain

$$x(t) = 2e^t.$$

Hence, the solution of the given Volterra integral equation is

$$x(t) = 2e^t.$$

Examples of Volterra Integral Equation of Second Kind

Example 5: Solve

$$U(x) = 1 - x - \int_0^x (x-t)u(t)\,dt$$

Solution:

Consider

$$U(x) = 1 - x - \int_0^x (x-t)u(t)\,dt$$

Taking $\mathfrak{J}$–transform to both sides, we have

$$\mathfrak{J}\{U(x)\} = \mathfrak{J}\left\{1 - x - \int_0^x (x-t)u(t)\,dt\right\}$$

$$\Rightarrow \mathfrak{J}\{U(x)\} = \mathfrak{J}\{1\} - \mathfrak{J}\{x\} - \mathfrak{J}\left\{\int_0^x (x-t)u(t)\,dt\right\}$$

$$\Rightarrow V(s,u) = \frac{u^2}{s} - \frac{u^3}{s^2} - \frac{1}{u}\left(\frac{u^3}{s^2}\right)V(s,u)$$

$$\Rightarrow V(s,u) + \frac{1}{u}\left(\frac{u^3}{s^2}\right)V(s,u) = \frac{su^2 - u^3}{s^2}$$

$$\Rightarrow V(s,u)\left(\frac{s^2+u^2}{s^2}\right) = \frac{su^2 - u^3}{s^2}$$

$$\Rightarrow V(s,u) = \frac{su^2 - u^3}{s^2+u^2}$$

Applying inverse $\mathfrak{J}$–transform,

$$U(x) = e^x$$

Example 6: Solve the Volterra integral equation of the second kind

$$x(t) = 1 + \int_0^t e^{t-\tau}x(\tau)\,d\tau.$$

Solution:
Let

$$k(t) = e^t.$$

Then, the given equation can be written in convolution form as

$$V(t) = 1 + (k * x)(t).$$

Applying the $\mathcal{J}$-transform to both sides, we obtain

$$V(s,u) = \mathcal{J}\{1\} + \mathcal{J}\{k * x\}.$$

Using the convolution theorem, we have

$$V(s,u) = \mathcal{J}\{1\} + \frac{1}{u}K(s,u)X(s,u),$$

where

$$K(s,u) = \mathcal{J}\{e^t\},$$

and

$$V(s,u) = \mathcal{J}\{x(t)\}.$$

First, we determine the transforms

$$\mathcal{J}\{1\} = \frac{u^2}{s},$$

and

$$K(s,u) = \mathcal{J}\{e^t\} = \frac{u^2}{s-u}.$$

Substituting these expressions into the transformed equation gives

$$V(s,u) = \frac{u^2}{s} + \frac{1}{u}\left(\frac{u^2}{s-u}\right)X(s,u).$$

Simplifying, we obtain

$$V(s,u) = \frac{u^2}{s} + \frac{u}{s-u}X(s,u).$$

Collecting the terms containing $X(s,u)$, we get

$$V(s,u) - \frac{u}{s-u}X(s,u) = \frac{u^2}{s}.$$

Hence,

$$V(s,u)\left(1 - \frac{u}{s-u}\right) = \frac{u^2}{s}.$$

Simplifying further,

$$V(s,u)\left(\frac{s-2u}{s-u}\right) = \frac{u^2}{s}.$$

Therefore,

$$V(s,u) = \frac{u^2(s-u)}{s(s-2u)}.$$

Using partial fractions, we have

$$V(s,u) = \frac{u^2}{2s} + \frac{u^2}{2(s-2u)}.$$

Taking the inverse $\mathcal{J}$-transform of both sides yields

$$x(t) = \frac{1}{2} + \frac{1}{2}e^{2t}.$$

Hence, the solution of the integral equation is

$$x(t) = \frac{1+e^{2t}}{2}.$$

Example 4: Solve the Volterra integral equation of the second kind

$$x(t) = t + \int_0^t (t-\tau)x(\tau)\,d\tau.$$

Solution:
Let

$$k(t) = t.$$

Then, the equation can be written in convolution form as

$$x(t) = t + (k * x)(t).$$

Applying the $\mathcal{J}$-transform to both sides, we obtain

$$V(s, u) = \mathcal{J}\{t\} + \mathcal{J}\{k * x\}.$$

Using the convolution theorem,

$$V(s, u) = \mathcal{J}\{t\} + \frac{1}{u} K(s, u) X(s, u),$$

where

$$K(s, u) = \mathcal{J}\{t\},$$

and

$$X(s, u) = \mathcal{J}\{x(t)\}.$$

Since

$$\mathcal{J}\{t\} = \frac{u^3}{s^2},$$

we have

$$V(s, u) = \frac{u^3}{s^2} + \frac{1}{u}\left(\frac{u^3}{s^2}\right) X(s, u).$$

Simplifying,

$$V(s, u) = \frac{u^3}{s^2} + \frac{u^2}{s^2} X(s, u).$$

Collecting the terms involving $X(s, u)$, we get

$$V(s, u)\left(1 - \frac{u^2}{s^2}\right) = \frac{u^3}{s^2}.$$

Hence,

$$V(s, u) = \frac{u^3}{s^2 - u^2}.$$

Factoring the denominator,

$$V(s, u) = \frac{u^3}{(s - u)(s + u)}.$$

Using partial fractions,

$$V(s, u) = \frac{u^2}{2(s - u)} - \frac{u^2}{2(s + u)}.$$

Taking the inverse $\mathcal{J}$-transform, we obtain

$$x(t) = \frac{1}{2} e^t - \frac{1}{2} e^{-t}.$$

Therefore,

$$x(t) = \sinh t.$$

CONCLUSION

In this paper, the J -transform has been applied to solve Volterra integral equations of both the first and second kind. By using the basic properties and convolution theorem of the transform, the integral equations were converted into algebraic equations in the transform domain, making the solution process more straightforward. The proposed procedure was illustrated through several examples, and in each case the exact solution was obtained successfully.

The results indicate that the J -transform can serve as an effective analytical tool for solving linear Volterra integral equations with convolution kernels. The method is easy to apply, requires relatively simple computations, and provides a systematic approach for obtaining exact solutions. This suggests that the J -transform can be considered as an alternative to other well-known integral transforms, such as the Laplace, Sumudu, Elzaki, and Natural transforms, for this class of problems.

The present study is limited to linear Volterra integral equations with convolution kernels. Future research may extend the proposed method to nonlinear Volterra integral equations, Volterra–Fredholm integral equations, fractional integral equations, integro-differential equations, and systems of integral equations. In addition, a detailed com-parison of the J -transform with existing integral transform methods in terms of compu-tational efficiency and applicability would provide further insight into its strengths and limitations.

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