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# Review on the Hunter–Saxton Equation: Integrable Structure, Singular Dynamics, and Fractional Generalizations

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## ABSTRACT

The Hunter–Saxton equation describes the propagation of orientation disturbances in nematic liquid crystals and is one of the most widely studied nonlinear hyperbolic variational equations. This review provides a comprehensive analysis of the published literature on the Hunter–Saxton equation, from its physical derivation to its modern fractional extensions. We first recall its origin in director-field dynamics and its surprising status as a completely integrable system. We then examine its bi-Hamiltonian structure, conservation laws, and Lie symmetries. A central theme is the geometric interpretation of the equation as a geodesic flow on the group of circle diffeomorphisms and its place within the family of peakon equations that includes Camassa–Holm and Degasperis–Procesi. We discuss singularity formation, wave breaking, blow-up criteria, and the modern theory of conservative and dissipative weak solutions. Numerical methods and discrete integrable schemes are surveyed. We then review the rapidly growing literature on fractional generalizations, including Caputo, Riemann–Liouville, conformable, and Atangana–Baleanu variants, and the exact Mittag-Leffler solutions obtained in 2026. We close by identifying open problems and future directions, including multi-dimensional extensions, stochastic and data-driven approaches, and the rigorous validation of fractional models against physical experiments.

**Keywords:** Hunter–Saxton Equation, Camassa–Holm Equation, Degasperis–Procesi Equation

## INTRODUCTION

The Hunter–Saxton equation was derived in 1991 as an asymptotic model for the dynamics of the director field in a nematic liquid crystal [1]. It describes the propagation of small-amplitude orientation waves under the combined influence of nonlinear convection and geometric nonlinearity. Smooth solutions of the equation can break down in finite time, even though the solution itself remains continuous [1]. This combination of nonlinear instability and wave-like propagation made the equation an attractive object for mathematical study.

A second decisive step came with the discovery that the Hunter–Saxton equation is completely integrable [2]. The authors exhibited an infinite family of conserved quantities and showed that the equation belongs to a hierarchy associated with the Korteweg–de Vries (KdV) equation. This placed the equation in a small and famous family of integrable nonlinear wave equations that also includes the Camassa–Holm equation, whose peaked solitons, or peakons, launched an enormous literature [3].

Since then, the Hunter–Saxton equation has been studied from many points of view. Analysts proved global existence of weak solutions for the underlying variational wave equation and studied the zero-viscosity and dispersion limits [4,5]. The formation of singularities was characterized in detail [6,7]. Geometers showed that the equation describes geodesic flow on an infinite-dimensional manifold of circle diffeomorphisms [8-10]. Numerical analysts developed dissipative, geometric, and integrable discretizations [11-13]. Most recently, fractional-order versions of the equation have attracted intense interest, with exact closed-form solutions reported in 2026 [14-16].

The purpose of this review is threefold. First, we provide a critical synthesis of the mathematical theory of the Hunter–Saxton equation. Second, we identify the trends that have shaped the field, from the geometric theory of peakon equations to the fractional calculus era. Third, we highlight open problems and promising directions for future research. Throughout, we emphasize the connections between the Hunter–Saxton equation and its better-known relatives, because these connections explain why the equation remains an active research topic.

For completeness, we note a point of terminology. The nonlinear variational wave equation studied by Hunter and Zheng is sometimes called the Hunter–Zheng equation, and the integrable asymptotic reduction that is the subject of this review is occasionally referred to by the same name [17]. Here we use the name Hunter–Saxton equation throughout, following the physical derivation of Hunter and Saxton [1].

The review is organized as follows. Section 2 describes the physical origin of the equation in nematic liquid crystals. Section 3 covers its integrable structure, symmetries, and conservation laws. Section 4 presents the geometric interpretation. Section 5 situates the equation within the Camassa–Holm family. Section 6 treats singular solutions, wave breaking, and blow-up. Section 7 reviews the theory of global weak solutions. Section 8 surveys numerical methods. Section 9 describes generalizations and multi-component systems. Section 10 reviews fractional Hunter–Saxton equations. Section 11 discusses open problems, and Section 12 concludes.

## Physical Origin: Director-Field Dynamics in Nematic Liquid Crystals

### Variational Derivation of The Hunter–Saxton Model

Nematic liquid crystals consist of elongated molecules that tend to align along a common direction. The local direction of alignment is described by a unit vector field, the director field [18,19]. Under an external disturbance, the orientation of the director can vary in space and time. When the elastic response is governed by a variational principle, weakly nonlinear orientation waves satisfy an asymptotic equation whose form depends on the geometry of the director field [1].

#### Breakdown of Smooth Solutions and Singularity Formation

Hunter and Saxton derived the equation that now bears their name by analyzing weakly nonlinear hyperbolic waves governed by variational principles [1]. For the director field of a nematic liquid crystal, they obtained the scalar equation

$$u_{xt} = u\,u_{xx} - (u_x)^2 ,$$

for the orientation angle u(x,t), together with the condition that smooth solutions break down in finite time. In the integrability literature the equation is usually written in the equivalent form

$$u_{xxt} + 2\,u_x\,u_{xx} + u\,u_{xxx} = 0 ,$$

which is the compatibility condition of a Lax pair [2].

### Relation to The Hunter–Zheng Variational Wave Equation

The variational-wave-equation program continued in a series of papers by Hunter and Zheng. They proved global existence of weak solutions for the general nonlinear hyperbolic variational wave equation and studied the zero-viscosity and dispersion limits [4,5]. Glassey, Hunter, and Zheng then gave a precise description of singularity formation, showing that the spatial derivative blows up while the solution remains continuous [6,7]. These results established the mathematical framework within which the Hunter–Saxton equation is still studied today. A related asymptotic equation, obtained from the variational wave equation by an integrating reduction, belongs to the Harry Dym hierarchy and carries its own bi-Hamiltonian structure [17].

Physical Relevance and Nematicon Structures in Nonlinear Optics

The physical relevance of these models has been revisited repeatedly. For the special case of equal splay and bend coefficients, Huang and Zheng proved that a system of variational wave equations modeling nematic liquid crystals has globally smooth solutions [20]. On the modeling side, the generalized Hunter–Saxton model continues to attract attention because it supports localized structures, called nematicons, that are relevant to nonlinear optics and reorientation phenomena in liquid crystals [21]. The exact form of the equation and the choice of nonlinearity parameter affect the geometry of these waves, including parabolic, cuspon, kink, and singular profiles [21].

## Integrability, Symmetries, and Conservation Laws

### Bi-Hamiltonian Structure and Lax Pair Formulation

The modern theory of integrable nonlinear wave equations rests on two ideas: hereditary symmetry operators and bi-Hamiltonian structures [22,23]. The Camassa–Holm equation and the Hunter–Saxton equation both belong to this framework, and both admit Lax pairs that encode their infinite families of conservation laws [2,3]. The Camassa–Holm equation [3]:

$$u_t + 2u_x - u_{xxt} + 3u\,u_x + 2\,u_x\,u_{xx} + u\,u_{xxx} = 0 .$$

For the Hunter–Saxton equation, Hunter and Zheng established integrability by exhibiting the equation as a member of a hierarchy of commuting flows [2]. A Lax representation with a spectral parameter, a bi-Hamiltonian formulation, and an infinite sequence of conserved densities were later obtained in the general framework of hereditary operators [22-24]. The conserved densities of the Hunter–Saxton equation have a remarkably rich structure, and their symmetry algebra has been described in detail [24]. In the inverse-scattering formulation, peakon solutions arise from soliton-type spectral data, and explicit multi-peakon formulas follow from the associated linear systems [25].

### Lie Symmetries, Multipliers, and Conservation Laws

Lie symmetry analysis provides a systematic route to exact solutions and conservation laws. Nadjafikhah and Ahangari performed the first complete Lie pointsymmetry classification of the Hunter–Saxton equation, identifying its full symmetry algebra and deriving conservation laws associated with its variational symmetries [26]. Their results revealed several nontrivial point symmetries, including scaling, projective, and translation symmetries, each of which generates invariant solution families.

Kakuli, Sinkala, and Masemola refined this picture by combining the multiplier method with the doublereduction technique, obtaining five nontrivial conservation laws, four of which correspond to Lie point symmetries [27]. The doublereduction method is particularly effective for equations whose characteristics exhibit degeneracy, and their work demonstrated that the Hunter–Saxton equation requires careful handling of multipliers to obtain physically meaningful conserved quantities.

In 2026, Kakuli revisited the symmetry structure from a variational perspective, showing that the multiplier determining system collapses to a single linear equation [28]. This reduction clarifies the Noether correspondence for the Hunter–Saxton equation: although the equation possesses infinitely many conserved densities due to its integrability, only a subset of its Lie point symmetries arise from Noether symmetries of a variational principle. This distinction is important because the standard form of the Hunter–Saxton equation is not strictly Euler–Lagrange; its variational formulation involves a degenerate Lagrangian whose Euler–Poincar structure complicates the symmetry–conservation relationship.

Beyond point symmetries, the Hunter–Saxton equation exhibits an infinite hierarchy of higher symmetries and cosymmetries. Morozov's construction of local recursion operators and higherorder cosymmetries for the generalized Hunter–Saxton equation [29] shows that the symmetry algebra contains nonlocal symmetries generated by the hereditary operator of the Hunter–Saxton hierarchy. These higher symmetries produce conserved densities of increasing differential order and encode the integrable structure of the equation, placing it alongside the Camassa–Holm and KdV equations within the broader hereditaryoperator framework [22–24].

The symmetry structure also interacts with the geometric interpretation of the equation. Because the Hunter–Saxton equation describes geodesic flow on the quotient of the diffeomorphism group of the circle by rotations [8–10], its Lie symmetries include transformations induced by the action of the diffeomorphism group. These geometric symmetries correspond to reparametrizations of characteristics and generate conservation laws related to geodesic invariants of the $H^1$ metric. In this sense, the Lie symmetry algebra reflects the underlying infinitedimensional geometry of the equation.

Recent work has extended symmetry analysis to fractional and variablecoefficient variants. Yu and Feng performed a full Lie symmetry and conservationlaw analysis for the timefractional Hunter–Saxton system with variable coefficients, showing that fractional derivatives preserve a deformed version of the classical symmetry algebra. Their results indicate that memory effects modify the scaling and projective symmetries but leave invariant a core set of solutiongenerating transformations [30]. Thomas and Bakkyaraj applied Lie symmetry methods to fractional Hunter–Saxton equations in the Caputo and Riemann–Liouville senses, obtaining invariant solutions and fractional conservation laws [31].

Taken together, these developments show that the symmetry structure of the Hunter–Saxton equation is both rich and subtle, and that the association between symmetries and conservation laws requires care. The equation possesses point symmetries, higher symmetries, nonlocal symmetries, and geometric symmetries arising from its interpretation as a geodesic flow. The combined Liesymmetry, multiplier, and hereditaryoperator approaches provide a complete picture of the conservationlaw structure and explain why the Hunter–Saxton equation supports such a large and diverse family of exact solutions.

### Higher Symmetries, Recursion Operators, and Dispersive Variants

The integrability of the generalized Hunter–Saxton equation was established by Morozov, who constructed a Lax representation with a non-removable spectral parameter, local recursion operators, an infinite-dimensional Lie algebra of higher symmetries, and infinitely many higher-order cosymmetries [29]. These structures were used to generate explicit globally defined solutions. For the dispersive Hunter–Saxton equation, Fei studied integrability and identified the conditions under which the dispersive term preserves the integrable structure [30].

Group-theoretic and algebro-geometric tools have also been applied. Bozhkov and Silva Junior performed a group analysis of the generalized Hunter–Saxton system [31]. Aratyn and co-workers derived rational solutions from Pad approximants, exploiting the connection between the equation and integrable hierarchies [32]. In three spatial dimensions, a generalized Hunter–Saxton equation introduced by Morozov was shown to fail the Painlevé test, indicating that integrability is delicate in higher dimensions [33].

## Geometry: Geodesic Flows and Diffeomorphism Groups

### Geodesic Interpretation on The $H^1$ Metric Space

One of the most beautiful aspects of the Hunter–Saxton equation is its geometric interpretation. The Camassa–Holm equation was recognized as a geodesic flow on the Bott–Virasoro group [34], and more generally on groups of diffeomorphisms equipped with a right-invariant metric [35,36]. The same framework applies to the Hunter–Saxton equation.

### Explicit Geodesics and Periodic Solutions

Lenells provided a rigorous foundation for this picture [8,9]. The Hunter–Saxton equation describes the geodesic flow of the H-dot-1 right-invariant metric on the quotient of the diffeomorphism group of the circle by the subgroup of rotations. Using the method of characteristics, he derived explicit formulas for the geodesics and obtained new explicit expressions for spatially periodic solutions [9]. In a striking special case, the Hunter–Saxton equation was shown to describe the geodesic flow on a sphere, which reduces the study of its solutions to classical rigid-body dynamics [37].

### Generalized Geometric Hunter–Saxton Equations

Khesin, Lenells, and Misiołek generalized these ideas to a family of geodesic equations on the group of circle diffeomorphisms, the generalized Hunter–Saxton equation [10]. They showed how the choice of metric controls the geometry of the flow and how the μ-Hunter–Saxton equation, in which the mean value of the solution is preserved, fits into the same framework [10]. The Euler equations on homogeneous spaces and Virasoro orbits provide the broader context [38].

### Connections to Optimal Transport and Diffeomorphism Factorization

Related geometric variants include the deformed Hunter–Saxton equation arising from geodesic flow on the Bott–Virasoro group [39], the modified Hunter–Saxton equation [40], and supersymmetric extensions obtained from the Harry Dym hierarchy [17,41,42]. More recently, Modin connected generalized Hunter–Saxton equations to optimal information transport and to the factorization of diffeomorphisms [43], opening a new channel between the geometric theory of these equations and optimal transport.

#### Connections to the Camassa–Holm family

The Hunter–Saxton equation cannot be understood in isolation. It sits at the center of a family of integrable equations with peaked solitons, and many of its structural properties mirror those of the Camassa–Holm equation.

### The Camassa–Holm Equation: Structure and Relevance

The Camassa–Holm equation was discovered in 1993 as an integrable shallow water equation with peaked solitons [3]. Its complete integrability, bi-Hamiltonian structure, and explicit peakon solutions were established shortly afterward [22,44]. The equation was later shown to be relevant to the water-wave problem in the shallow-water regime [45,46], and its hydrodynamical derivation was refined through variational principles [47-49]. Family members with linear and nonlinear dispersion were found by Dullin, Gottwald, and Holm [50], and the asymptotic equivalence of these equations was analyzed in detail [51,52].

The Hunter–Saxton equation is, in a precise sense, the high-frequency limit of the Camassa–Holm equation, obtained by dropping the dispersion term [46,53]. This limiting relation explains the deep parallels between the two equations: both are integrable, both describe geodesic flows on diffeomorphism groups, and both possess peakon solutions [3,9,25]. The traveling wave structure of both equations was classified by Lenells [53]. On the circle, the Camassa–Holm equation was analyzed by Constantin and McKean, who solved the associated spectral problem and described the corresponding dynamics [54]. For initial data on the line, Boutet de Monvel and Shepelsky developed a Riemann–Hilbert formalism that represents the solution in parametric form and shows that the large-time behavior consists of trains of solitons that converge to peakons in the zero-dispersion limit [55].

### Degasperis–Procesi and Cubic Peakon Equations

The Degasperis–Procesi equation, found in 2002, is a second integrable equation with peakon solutions [56]. Unlike the Camassa–Holm equation, it admits shock-peakons and has a different conservation-law structure. The cubic integrable peakon equations discovered by Hone and Wang [57], and the Novikov equation [58], complete the family of integrable peakon equations. Qiao found equations with cuspons and W/M-shape peakons [59]. These equations share the geometric origin of the Camassa–Holm equation but display genuinely new singular solutions, and they have become benchmarks for the theory of weak solutions [60]. Every member of the family admits a Lax pair and a bi-Hamiltonian structure, and each exhibits its own family of singular traveling waves, making the family an ideal laboratory for testing general theories of integrable peakon equations [57-59].

### Well-Posedness and Blow-Up Phenomena Across the Family

The Camassa–Holm equation and its relatives have driven the development of well-posedness theory for equations with nonlocal nonlinearities. Constantin and Escher proved wave breaking for the Camassa–Holm equation and established sharp blow-up criteria [61,62]. Constantin gave a geometric approach to permanent and breaking waves [63], and McKean characterized the breakdown of solutions [64]. Global weak solutions were constructed by Constantin and Escher [65] and, in the integrable periodic case, by Yin [66]. Li and Olver obtained well-posedness and blow-up results for a family of integrable nonlinearly dispersive equations [67].

Two-component generalizations of the Camassa–Holm equation were introduced by Constantin and Ivanov [68] and studied in depth by Guan and Yin [69]. The lessons learned from this family, especially the correct treatment of peakons and wave breaking, transferred directly to the Hunter–Saxton equation [70,71].

#### Singular Solutions, Wave Breaking, and Blow-Up

Peakons, Cuspons, and Singular Traveling Waves

The Hunter–Saxton equation admits peaked traveling waves. Beals, Sattinger, and Szmigielski solved the inverse scattering problem for the equation and obtained explicit peakon solutions [25]. These solutions are continuous with a corner at their crest, and they interact elastically. The geometric approach of Lenells produced additional families of explicit solutions, including cusp-type waves [9,37]. In the Camassa–Holm family, ghostpeakons and characteristic curves were classified by Lundmark and Shuaib [60], and Qiao constructed cuspons and W/M-shape peakons for integrable equations related to the Hunter–Saxton equation [59]. For periodic data, a Riemann–Hilbert formalism for the Camassa–Holm equation was presented in 2026 [72], providing spectral data determined by the initial condition; similar tools are expected to clarify the periodic spectral theory of the Hunter–Saxton equation.

**Wave Breaking and Blow-Up Criteria**

The generic phenomenon in the Hunter–Saxton equation is wave breaking: the solution remains continuous while its spatial derivative blows up in finite time [1,6]. This is the same behavior as in the Camassa–Holm equation [61,64]. For variants of the equation, sharp blow-up criteria have been established. Li and Yin proved blow-up and traveling wave results for the periodic integrable dispersive Hunter–Saxton equation [73], and Jiang obtained a finite-time blow-up result for periodic solutions of the dispersive Hunter–Saxton equation using conserved quantities and the Gagliardo–Nirenberg inequality [74]. Wei and Yin established global existence and blow-up for the periodic Hunter–Saxton equation with weak dissipation [75], and Guo and Xiong analyzed blow-up for a periodic two-component μ-Hunter–Saxton system [76].

**Local and Global WellPosedness in Various Function Spaces**

The local and global well-posedness of the Hunter–Saxton equation has been studied in many function spaces. For the modified Hunter–Saxton equation, well-posedness was proved in Besov spaces by Mi and Mu [77] and, later, on the Cauchy problem, by Mi, Mu, and Zheng [78]. The periodic modified equation was treated by Tiğlay [79]. Nonuniform dependence on the initial data for the periodic Hunter–Saxton equation was established by Holliman [80]. For the dispersive Hunter–Saxton equation, Ai and Avadanei proved local and global well-posedness using normal-form energies and frequency envelopes, overcoming the lack of L-squared control that makes the equation quasilinear [81]. Initial-boundary-value problems for variants of the equation were analyzed by Kohlmann [82], and weakly dissipative versions of the Camassa–Holm, Degasperis–Procesi, and Novikov equations were studied by Lenells and Wunsch [83].

**Global Weak Solutions: Conservative and Dissipative**

Because smooth solutions break down, a satisfactory theory requires weak solutions defined beyond the time of wave breaking. The Hunter–Saxton equation sits within a hierarchy of variational wave equations for which two canonical solution classes have been developed: conservative and dissipative. The distinction is visible already at the level of the energy. Conservative solutions preserve the energy functional for all time, whereas dissipative solutions lose energy exactly at the breaking times [84,85].

**Weak Solutions for Variational Wave Equations**

Hunter and Zheng proved the global existence of weak solutions for the nonlinear hyperbolic variational wave equation [4] and analyzed the zero-viscosity and dispersion limits [5]. Zhang and Zheng proved existence and uniqueness for the asymptotic equation [86], and Zhang developed the theory of weak solutions in a systematic way [87]. Glassey, Hunter, and Zheng described the singularities that can occur [6,7]. This body of work established that energy may concentrate on sets of measure zero, which is the origin of the distinction between conservative and dissipative solutions.

**Conservative Solutions and Uniqueness Theory**

Conservative solutions preserve the energy and were constructed for the nonlinear variational wave equation by Bressan and Zheng [84] and, in a semigroup setting, by Holden and Raynaud [88]. Bressan, Chen, and Zhang proved uniqueness of conservative solutions using characteristics [89] and extended the method to the Camassa–Holm equation [90]. Hu constructed conservative solutions for systems of variational wave equations [91-93], for the one-dimensional nonlinear variational wave equation [94], and for a nonlinear variational sine-Gordon equation [95]. For the Hunter–Saxton equation itself, Grunert and Holden proved the uniqueness of conservative solutions [96], completing the program for this equation.

**Dissipative Solutions and Maximal Dissipation**

Dissipative solutions lose energy through wave breaking. The dissipative theory for the Camassa–Holm equation was developed by Bressan and Constantin [85] and by Holden and Raynaud [97]. For the variational wave equation, Dafermos used generalized characteristics to prove the uniqueness of dissipative solutions [71], and Bressan and Huang gave a representation of dissipative solutions [98]. The maximal-dissipation conjecture of Zhang and Zheng was proved in full generality by Cieślak and Jamróz [99]. These results show that dissipative solutions are canonical in a precise sense: they dissipate energy at the maximal possible rate.

**Lipschitz Metrics and Stability Theory**

A persistent difficulty is that the solution map for weak solutions is not continuous in the natural norms. This problem was solved by the introduction of Lipschitz metrics. Bressan, Holden, and Raynaud constructed a Lipschitz metric for the Hunter–Saxton equation [100], and Carrillo, Grunert, and Holden built a Wasserstein-based Lipschitz metric [101]. Grunert and Tandy established

Lipschitz stability in time [102] and constructed Lipschitz metrics for α-dissipative solutions [103]. These developments make it possible to study perturbations of weak solutions in a meaningful way.

### Periodic and Weakly Dissipative Variants

The theory has been extended to periodic settings and to equations with weak dissipation. For the periodic Hunter–Saxton equation with weak dissipation, Wei proved global existence of weak solutions [104] and Wei and Yin obtained global existence and blow-up results [75]. Wei also constructed global weak solutions for the periodic generalized Hunter–Saxton equation [105]. Liu treated the weakly dissipative μ-Hunter–Saxton equation [106], and Wang, Li, and Chen studied the Cauchy problem for the weakly dissipative generalized μ-Hunter–Saxton equation [107].

## Numerical Methods

### SemiAnalytical and Spectral Approaches

The numerical study of the Hunter–Saxton equation began with semi-analytical and spectral approaches. Arbabi, Nazari, and Darvishi obtained semi-analytical solutions [108], Parand and Delkhosh developed a hybrid quasilinearization–collocation method using bivariate generalized fractional-order Chebyshev functions [109], and Hashmi and co-workers applied a cubic trigonometric B-spline collocation method [110]. Nirmala and Kumbinarasaiah used the Hosoya polynomial method to solve the nonlinear Hunter–Saxton equation together with other evolution equations [111].

### Dissipative Numerical Schemes for Weak Solutions

For weak solutions, Xu and Shu constructed dissipative numerical methods based on discontinuous Galerkin and Runge–Kutta techniques [11]. A more structural approach was taken by Penskoi, who obtained Lagrangian time discretizations of the equation using the Moser–Veselov method [12], and by Ivanov, who developed algebraic discretizations [112]. Miyatake, Cohen, Furihata, and Matsuo introduced geometric numerical integrators for Hunter–Saxton-like equations based on new multi-symplectic formulations and the discrete variational derivative method [13]. The choice of dissipation model strongly affects the numerical result: conservative schemes must track the energy measure, whereas dissipative schemes must enforce energy loss at wave breaking [11,113]. Combining geometric integrators with the Lipschitz-metric stability theory is a promising route to rigorous a posteriori error estimates [13,102].

### Geometric and Integrable Discretizations

In 2026, Christiansen proved convergence-rate estimates for a numerical method for α-dissipative solutions of the Hunter–Saxton equation, obtaining explicit rates in the bounded-Lipschitz metric [113]. This work closes a gap between the numerical and analytical theories of weak solutions.

### Particle Methods and LongTime Integrable Algorithms

The particle-method philosophy developed for the Camassa–Holm equation has also informed the numerical treatment of the Hunter–Saxton equation. Camassa, Huang, and Lee constructed completely integrable numerical schemes [114], integral and integrable algorithms [115], and complete integrable particle methods that exhibit recurrence of initial states [116]. These methods respect the integrable structure of the equations and provide highly accurate long-time computations.

## Generalizations and Multi-Component Systems

### Two-Component Systems

The Hunter–Saxton equation has been extended to two-component systems in several ways. Wunsch introduced and analyzed the Hunter–Saxton system [117], derived its generalized form [118], and studied its weak geodesic flow on a semidirect product [119]. Zuo proposed a two-component μ-Hunter–Saxton equation and showed that it is a bi-Hamiltonian Euler equation [120]. Hou and Fan obtained algebro-geometric solutions for the two-component Hunter–Saxton hierarchy [121]. Li, Wen, and Chen found single-peak and compacton solutions of the generalized two-component system [122]. Most recently, Maier studied the magnetic two-component Hunter–Saxton system, establishing global conservative weak solutions through a relaxed configuration space [123].

### Modified, Higher-Order, and Geometric Variants

The modified Hunter–Saxton equation [40] has received renewed attention. Xie, Tian, and Yan developed a generalized Fourier transform for the modified Hunter–Saxton equation [124], and Bayrakdar and Bayrakdar gave a moving-curve description of the Hunter–Saxton equation [125]. Cotter, Deasy, and Pryer introduced the r-Hunter–Saxton equation, arising as the extremal of an action principle in L_r, and characterized its singular solutions [126]. Mixed equations that interpolate between the Camassa–Holm and Hunter–Saxton equations were studied by Zhang and Hu [127]. These variants illustrate a general pattern: each integrable member of the family produces its own singular structures and requires its own well-posedness theory [40,124,126].

### Stochastic Hunter–Saxton Equations

A recent and active direction is the inclusion of noise. Holden, Karlsen, and Pang developed an existence theory for the stochastic Hunter–Saxton equation and proved several properties of its wave-breaking [128]. In the linear-noise case, they used stochastic characteristics to derive an explicit law for the random wave-breaking time. Stochastic versions of these equations raise entirely new questions about singularity formation and are a promising area for future work.

#### Fractional Hunter–Saxton Equations

#### Fractional Calculus Background

Fractional-order operators provide a natural framework for modeling memory effects [129,130]. The Caputo derivative, introduced in the study of dissipation in geophysics [131-133], is particularly suited to physical models because it allows standard initial conditions. The Mittag-Leffler function plays the role of the exponential function in this calculus [134,135]. Non-singular kernels, such as the Atangana–Baleanu derivative, were introduced to reduce the singular behavior of the classical kernels [136].

### Time-Fractional Hunter–Saxton Equations

A time-fractional extension of the Hunter–Saxton equation was first proposed by Atangana, Baleanu, and Alsaedi, who analyzed the model using the homotopy decomposition method and established stability in a Hilbert-space setting [137]. Gmez-Aguilar and Atangana studied the fractional Hunter–Saxton equation with partial operators of bi-order in the Riemann–Liouville and Liouville–Caputo senses [138]. Lie symmetry analysis was applied to the fractional equation by Thomas and Bakkyaraj [139], and Yu and Feng performed a full Lie symmetry and conservation-law analysis for the time-fractional Hunter–Saxton system with variable coefficients [140].

### Exact Fractional Solutions and 2026 Developments

In 2026, two analytical approaches produced exact solutions. Sagar and Mohapatra used a separation-of-variables approach based on the conformable derivative to obtain new traveling-wave solutions of the time-fractional generalized Hunter–Saxton model in nematic liquid crystals [15]. Alleddawi, Momani, and Az-Zo'bi introduced a fourth-order dispersive extension of the generalized Hunter–Saxton equation and studied the effects of several fractional derivatives on traveling-wave reductions [16].

A key development in 2026 was the first exact closed-form solution of the time-fractional Hunter–Saxton equation with a Caputo derivative [14]. The method exploits the exponential spatial profile exp(b+x), whose derivative identities cause the nonlinear quadratic terms to cancel. The equation then reduces to a fractional relaxation ODE, whose solution is the one-parameter Mittag-Leffler function [14]. As the fractional order $\alpha$ tends to 1, this solution converges to the classical exponential traveling wave of the Hunter–Saxton equation. This result provides a benchmark for numerical schemes and clarifies the role of memory in orientation-wave dynamics [14]. It also highlights a general principle: for carefully chosen spatial modes, nonlinear fractional PDEs can be reduced to linear fractional ODEs, and this principle is expected to apply to other integrable systems.

The choice of fractional operator is not cosmetic. The conformable derivative is a local operator that obeys a modified chain rule, so conformable models reduce to classical ODEs under a traveling-wave ansatz [15]. The Caputo derivative, by contrast, is a nonlocal convolution operator whose kernel couples the present state to the entire past history of the solution. For negative coefficients, solutions of the Caputo model relax algebraically rather than exponentially, and the Mittag-Leffler function interpolates between algebraic and exponential behavior as the order $\alpha$ varies between 0 and 1 [14,134,135]. These differences matter for physical interpretation: memory-driven, slower-than-exponential relaxation is exactly the signature that distinguishes fractional from classical director-field dynamics [14,21].

### Fractional Camassa–Holm and Related Equations

The fractionalization program extends to the whole Camassa–Holm family. Saha Ray and Sahoo constructed traveling-wave solutions of the Riesz time-fractional Camassa–Holm equation [141], and Gupta, Singh, and Yildirim obtained approximate analytical solutions of time-fractional Camassa–Holm, modified Camassa–Holm, and Degasperis–Procesi equations by homotopy perturbation [142]. Kumar, Singh, and Baleanu analyzed the Fornberg–Whitham equation with a Mittag-Leffler-type kernel [143]. These studies show that the tools developed for the Hunter–Saxton equation transfer directly to its relatives.

## OPEN PROBLEMS AND FUTURE DIRECTIONS

Despite the depth of the theory, several important problems remain open.

#### Exact Solution

First, the exact solution of the Caputo time-fractional Hunter–Saxton equation found in 2026 [14] relies on a privileged exponential spatial profile. Whether other profiles, or genuinely general solutions, can be obtained remains an open question. A rigorous theory of well-posedness for the fractional equation, including uniqueness and continuous dependence, is still missing. The conformable-derivative solutions [15] and the local Caputo solution [14] must be reconciled, because only the Caputo and Riemann–Liouville operators capture genuine nonlocal memory [14,138].

Second, the interplay between fractional dynamics and wave breaking is poorly understood. It is not known how fractional memory changes the blow-up time or whether conservative and dissipative solutions can be defined for fractional equations. The stochastic Hunter–Saxton equation [128] raises similar questions in the presence of noise.

#### MultiDimensional and Geometric Extensions

Third, the multi-dimensional and multi-component theory is still developing. The magnetic two-component system [123], the r-Hunter–Saxton equation [126], and three-dimensional variants [33] show that integrability is delicate in higher dimensions. Whether geometrically meaningful multi-dimensional extensions exist remains an open problem.

#### Stochastic and DataDriven Approaches

Fourth, data-driven methods have not yet been applied to the Hunter–Saxton equation. Machine-learning approaches could help identify hidden conservation laws, discover exact solutions of fractional variants, and calibrate fractional orders against experimental measurements of reorientation waves in liquid crystals.

Fifth, the Riemann–Hilbert method has proved powerful for the Camassa–Holm equation on the line [55] and in the periodic case [72], but its adaptation to the Hunter–Saxton equation is still incomplete. A complete spectral theory for the periodic problem would likely settle the long-time asymptotics of peakon and cuspon trains and provide the rigorous underpinning for the inverse-scattering solutions constructed by Beals, Sattinger, and Szmigielski [25].

#### Physical Validation of Fractional Models

Finally, experimental validation is a priority. The fractional models predict slower, memory-driven relaxation of director-field disturbances [14,21]. Testing these predictions against observations of reorientation phenomena in nematic liquid crystals would firmly establish the physical relevance of the fractional theory.

## CONCLUSION

The Hunter–Saxton equation has matured from an asymptotic model of director-field dynamics in liquid crystals into a cornerstone of the theory of integrable nonlinear wave equations. Its bi-Hamiltonian structure, conservation laws, and Lie symmetries are now well understood [24,26,29]. Its geometric interpretation as a geodesic flow on the group of circle diffeomorphisms connects it to the theory of peakon equations and to optimal transport [9,10,43]. Its singular solutions exhibit the full range of wave-breaking phenomena, and its conservative and dissipative weak solutions have been characterized with remarkable precision [70,96,99]. Numerical methods now respect the integrable and geometric structure of the equation [13,113].

The most exciting recent development is the fractional program. The exact Mittag-Leffler solution of the Caputo time-fractional Hunter–Saxton equation [14] and the traveling-wave solutions of conformable and higher-order variants [15,16] have opened a new chapter in the study of this equation. These results bridge the classical and fractional worlds and provide benchmarks for numerical schemes. At the same time, they raise fundamental questions about uniqueness, well-posedness, and physical relevance that will drive the field forward.

We conclude that the Hunter–Saxton equation remains a remarkably productive tested for ideas in nonlinear PDE theory, integrable systems, differential geometry, numerical analysis, and fractional calculus. Its future will likely be shaped by the convergence of these fields: fractional-order models, stochastic perturbations, data-driven discovery, and experimental verification.

## DECLARATIONS

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## REFERENCES

1. Hunter JK, Saxton R. Dynamics of director fields. SIAM J Appl Math. 1991;51(6):1498-1521.
2. Hunter JK, Zheng Y. On a completely integrable nonlinear hyperbolic variational equation. Physica D. 1994;79(2-4):361-386.
3. Camassa R, Holm DD. An integrable shallow water equation with peaked solitons. Phys Rev Lett. 1993;71(11):1661-1664.
4. Hunter JK, Zheng Y. On a nonlinear hyperbolic variational equation: I. Global existence of weak solutions. Arch Ration Mech Anal. 1995;129(4):305-353.
5. Hunter JK, Zheng Y. On a nonlinear hyperbolic variational equation: II. The zero-viscosity and dispersion limits. Arch Ration Mech Anal. 1995;129(4):355-383.
6. Glassey RT, Hunter JK, Zheng Y. Singularities of a variational wave equation. J Differential Equations. 1996;129(1):49-78.
7. Glassey RT, Hunter JK, Zheng Y. Singularities and oscillations in a nonlinear variational wave equation. The IMA Volumes in Mathematics and its Applications. 1997;100:37-60.
8. Lenells J. Weak geodesic flow and global solutions of the Hunter-Saxton equation. Discrete and Continuous Dynamical Systems. 2007;18(4):643-656.
9. Lenells J. The Hunter-Saxton equation: a geometric approach. SIAM J Math Anal. 2008;40(1):266-277.

10. Khesin B, Lenells J, Misiołek G. Generalized Hunter-Saxton equation and the geometry of the group of circle diffeomorphisms. Math Ann. 2008;342(3):617-656.
11. Xu Y, Shu CW. Dissipative numerical methods for the Hunter-Saxton equation. J Comput Math. 2010;28(5):606-620.
12. Penskoi AV. Lagrangian time-discretization of the Hunter-Saxton equation. Phys Lett A. 2002;304(5-6):157-167.
13. Miyatake Y, Cohen D, Furihata D, Matsuo T. Geometric numerical integrators for Hunter-Saxton-like equations. Japan Journal of Industrial and Applied Mathematics. 2017;34(2):441-472.
14. Huang W. Exact analytic solution for the time-fractional Hunter-Saxton equation with Caputo derivative. Nonlinear Science. 2026;8(10):100193.
15. Sagar B, Mohapatra RN. New traveling-wave solutions for time-fractional generalized Hunter-Saxton model in nematic liquid crystals: a separation of variables approach. Europhysics Letters. 2026;153(2):22001.
16. Alleddawi A, Momani S, Az-Zo'bi EA. A novel higher-order dispersive extension of the generalized Hunter-Saxton model: traveling-wave solutions, time-fractional impacts, and qualitative analysis. Modern Physics Letters B. 2026;40(15):2650105.
17. Brunelli JC, Das A, Popowicz Z. Supersymmetric extensions of the Harry Dym hierarchy. J Math Phys. 2003;44(10):4756-4767.
18. de Gennes PG, Prost J. The physics of liquid crystals. 2nd ed. Oxford: Clarendon Press; 1993.
19. Stewart IW. The static and dynamic continuum theory of liquid crystals: a mathematical introduction. London: Taylor & Francis; 2004.
20. Huang J, Zheng Y. No blow-up to a variational wave equation in liquid crystals. J Math Phys. 2016;57(2):021506.
21. Az-Zo'bi EA. Bifurcation geometry, global stability, and nonlinear nematicon dynamics of the generalized Hunter-Saxton model. Mathematics. 2026;14(1):142.
22. Fuchssteiner B, Fokas AS. Symplectic structures, their Bäcklund transformations and hereditary symmetries. Physica D. 1981;4(1):47-66.
23. Fokas AS, Fuchssteiner B. Bäcklund transformations for hereditary symmetries. Nonlinear Anal. 1981;5(4):423-432.
24. Wang JP. The Hunter-Saxton equation: remarkable structures of symmetries and conserved densities. Nonlinearity. 2010;23(8):2009-2028.
25. Beals R, Sattinger DH, Szmigielski J. Inverse scattering solutions of the Hunter-Saxton equation. Applicable Analysis. 2001;78(3-4):255-269.
26. Nadjafikhah M, Ahangari F. Symmetry analysis and conservation laws for the Hunter-Saxton equation. Communications in Theoretical Physics. 2013;59(3):335-348.
27. Kakuli MC, Sinkala W, Masemola P. Conservation laws and symmetry reductions of the Hunter-Saxton equation via the double reduction method. Preprints.org Preprint. 2023;202307.0787.v1.
28. Kakuli MC. First-order multipliers, Noether symmetries and variational reduction of the Hunter-Saxton equation. Preprints.org Preprint. 2026;202608.0328.v1.
29. Morozov OI. Integrability structures of the generalized Hunter-Saxton equation. Analysis and Mathematical Physics. 2021;11(2):57.
30. Fei M. The integrability of dispersive Hunter-Saxton equation. Journal of Partial Differential Equations. 2012;25(4):330-334.
31. Bozhkov Y, Silva Junior VA. Group analysis of the generalized Hunter-Saxton system. Discontinuity, Nonlinearity, and Complexity. 2017;6(2):165-171.
32. Aratyn H, Gomes JF, Ruy DV, Zimerman AH. Rational solutions from Padé approximants for the generalized Hunter-Saxton equation. Journal of Physics: Conference Series. 2013;474:012006.
33. Sakovich S. On a new 3D generalized Hunter-Saxton equation. Nonlinear Phenom Complex Syst. 2025;28(1):64-67. DOI
34. Misiołek G. A shallow water equation as a geodesic flow on the Bott-Virasoro group. J Geom Phys. 1998;24(3):203-208.
35. Holm DD, Marsden JE, Ratiu TS. Euler-Poincaré models of ideal fluids with nonlinear dispersion. Phys Rev Lett. 1998;80(19):4173-4176.
36. Holm DD, Marsden JE, Ratiu TS. The Euler-Poincaré equations and semidirect products with applications to continuum theories. Adv Math. 1998;137(1):1-81.
37. Lenells J. The Hunter-Saxton equation describes the geodesic flow on a sphere. J Geom Phys. 2007;57(10):2049-2064.
38. Khesin B, Misiołek G. Euler equations on homogeneous spaces and Virasoro orbits. Adv Math. 2003;176(1):116-144.
39. Guha P. Geodesic flow on the Bott-Virasoro group and deformed Hunter-Saxton equation. Inverse Problems. 2004;20(5):1479-1484.
40. Górka P, Reyes EG. The modified Hunter-Saxton equation. J Geom Phys. 2012;62(8):1793-1809.
41. Liu QP, Popowicz Z, Tian K. The even and odd supersymmetric Hunter-Saxton and Liouville equations. Phys Lett A. 2010;375(1):29-35.
42. Lenells J, Lechtenfeld O. On the N=2 supersymmetric Camassa-Holm and Hunter-Saxton equations. J Math Phys. 2009;50(1):012704.
43. Modin K. Generalized Hunter-Saxton equations, optimal information transport, and factorization of diffeomorphisms. The Journal of Geometric Analysis. 2015;25(2):1306-1334.
44. Camassa R, Holm DD, Hyman JM. A new integrable shallow water equation. Advances in Applied Mechanics. 1994;31:1-33.
45. Johnson RS. Camassa-Holm, Korteweg-de Vries and related models for water waves. Journal of Fluid Mechanics. 2002;455:63-82.
46. Constantin A, Lannes D. The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations. Arch Ration Mech Anal. 2009;192(1):165-186.
47. Ionescu-Kruse D. Variational derivation of the Camassa-Holm shallow water equation. J Nonlinear Math Phys. 2007;14(3):311-318.
48. Ionescu-Kruse D. Variational derivation of the Camassa-Holm shallow water equation with non-zero vorticity. Discrete and Continuous Dynamical Systems. 2007;19(3):531-543.
49. Ionescu-Kruse D. Variational derivation of a geophysical Camassa-Holm type shallow water equation. Nonlinear Anal. 2017;156:286-294.
50. Dullin HR, Gottwald GA, Holm DD. An integrable shallow water equation with linear and nonlinear dispersion. Phys Rev Lett. 2001;87(19):194501.
51. Dullin HR, Gottwald GA, Holm DD. Camassa-Holm, Korteweg-de Vries-5 and other asymptotically equivalent equations for shallow water waves. Fluid Dynamics Research. 2003;33(1-2):73-95.

52. Dullin HR, Gottwald GA, Holm DD. On asymptotically equivalent shallow water wave equations. Physica D. 2004;190(1-2):1-14.
53. Lenells J. Traveling wave solutions of the Camassa-Holm and Korteweg-de Vries equations. J Nonlinear Math Phys. 2004;11(4):508-520.
54. Constantin A, McKean HP. A shallow water equation on the circle. Commun Pure Appl Math. 1999;52(8):949-982.
55. Boutet de Monvel A, Shepelsky D. Riemann-Hilbert approach for the Camassa-Holm equation on the line. Comptes Rendus Mathematique. 2006;343(10):627-632.
56. Degasperis A, Holm DD, Hone ANW. A new integrable equation with peakon solutions. Theoretical and Mathematical Physics. 2002;133(2): 1463-1474.
57. Hone ANW, Wang JP. Integrable peakon equations with cubic nonlinearity. J Phys A Math Theor. 2008;41(37):372002.
58. Novikov V. Generalizations of the Camassa-Holm equation. J Phys A Math Theor. 2009;42(34):342002.
59. Qiao Z. A new integrable equation with cuspons and W/M-shape-peaks solitons. J Math Phys. 2006;47(11):112701.
60. Lundmark H, Shuaib B. Ghostpeakons and characteristic curves for the Camassa-Holm, Degasperis-Procesi and Novikov equations. SIGMA. 2019;15:017.
61. Constantin A, Escher J. Wave breaking for nonlinear nonlocal shallow water equations. Acta Mathematica. 1998;181(2):229-243.
62. Constantin A, Escher J. Well-posedness, global existence, and blowup phenomena for a periodic quasi-linear hyperbolic equation. Commun Pure Appl Math. 1998;51(5):475-504.
63. Constantin A. Existence of permanent and breaking waves for a shallow water equation: a geometric approach. Annales de l'Institut Fourier. 2000;50(2):321-362.
64. McKean HP. Breakdown of a shallow water equation. Asian Journal of Mathematics. 1998;2(4):867-874.
65. Constantin A, Escher J. Global weak solutions for a shallow water equation. Indiana University Mathematics Journal. 1998;47(4):1527-1545.
66. Yin Z. Global weak solutions for a new periodic integrable equation with peakon solutions. J Funct Anal. 2004;212(1):182-194.
67. Li YA, Olver PJ. Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation. J Differential Equations. 2000;162(1):27-63.
68. Constantin A, Ivanov RI. On an integrable two-component Camassa-Holm shallow water system. Phys Lett A. 2008;372(48):7129-7132.
69. Guan C, Yin Z. Global existence and blow-up phenomena for an integrable two-component Camassa-Holm shallow water system. J Differential Equations. 2010;248(8):2003-2014.
70. Bressan A, Constantin A. Global conservative solutions of the Camassa-Holm equation. Arch Ration Mech Anal. 2007;183(2):215-239.
71. Dafermos CM. Generalized characteristics and the Hunter-Saxton equation. J Hyperbol Differ Equ. 2011;8(1):159-168.
72. Boutet de Monvel A, Karpenko I, Shepelsky D, Zielinski L. The periodic Camassa-Holm equation by the Riemann-Hilbert problem approach. SSRN Preprint. 2026;6659874.
73. Li M, Yin Z. Blow-up phenomena and travelling wave solutions to the periodic integrable dispersive Hunter-Saxton equation. Discrete and Continuous Dynamical Systems. 2017;37(12):6471-6485.
74. Jiang B. A blow-up result for the periodic solutions to an integrable dispersive Hunter-Saxton equation. J Nonlinear Math Phys. 2022;29(1):115-123.
75. Wei X, Yin Z. Global existence and blow-up phenomena for the periodic Hunter-Saxton equation with weak dissipation. J Nonlinear Math Phys. 2011;18(1):139-149.
76. Guo Y, Xiong T. Blow-up analysis for a periodic two-component mu-Hunter-Saxton system. Journal of Inequalities and Applications. 2018;2018(1): 310.
77. Mi Y, Mu C. Well-posedness for the Cauchy problem of the modified Hunter-Saxton equation in the Besov spaces. Mathematical Methods in the Applied Sciences. 2014;38(17):4061-4074.
78. Mi Y, Mu C, Zheng P. On the Cauchy problem of the modified Hunter-Saxton equation. Discrete and Continuous Dynamical Systems - S. 2016;9(6):2047-2072.
79. Tiğlay F. The periodic Cauchy problem of the modified Hunter-Saxton equation. Journal of Evolution Equations. 2005;5(4):509-527.
80. Holliman C. Nonuniform dependence and well posedness for the periodic Hunter-Saxton equation. Differential and Integral Equations. 2010;23(11-12):1151-1176.
81. Ai A, Avadanei ON. Well-posedness for the dispersive Hunter-Saxton equation. Int Math Res Not. 2023;2023(9):7883-7924.
82. Kohlmann M. On initial boundary value problems for variants of the Hunter-Saxton equation. Zeitschrift für angewandte Mathematik und Physik. 2011;62(5):855-878.
83. Lenells J, Wunsch M. On the weakly dissipative Camassa-Holm, Degasperis-Procesi, and Novikov equations. J Differential Equations. 2013;255(3): 441-448.
84. Bressan A, Zheng Y. Conservative solutions to a nonlinear variational wave equation. Commun Math Phys. 2006;266(2):471-497.
85. Bressan A, Constantin A. Global dissipative solutions of the Camassa-Holm equation. Analysis and Applications. 2007;5(1):1-27.
86. Zhang P, Zheng Y. On the existence and uniqueness of solutions to an asymptotic equation of a variational wave equation. Acta Mathematica Sinica, English Series. 1999;15(1):115-129.
87. Zhang P. Weak solutions to a nonlinear variational wave equation and some related problems. Applications of Mathematics. 2006;51(4):427-466.
88. Holden H, Raynaud X. Global semigroup of conservative solutions of the nonlinear variational wave equation. Arch Ration Mech Anal. 2011;201(3):871-964.
89. Bressan A, Chen G, Zhang Q. Unique conservative solutions to a variational wave equation. Arch Ration Mech Anal. 2015;217(3):1069-1101.
90. Bressan A, Chen G, Zhang Q. Uniqueness of conservative solutions to the Camassa-Holm equation via characteristics. Discrete and Continuous Dynamical Systems. 2015;35(1):25-42.
91. Hu Y. Conservative solutions to a system of variational wave equations. J Differential Equations. 2012;252(6):4002-4026.

92. Hu Y. Global energy conservative solutions to a system of variational wave equations. Nonlinear Anal. 2012;75(18):6418-6432.
93. Hu Y. Conservative solutions to a system of asymptotic variational wave equations. J Math Anal Appl. 2013;408(2):725-732.
94. Hu Y. Conservative solutions to a one-dimensional nonlinear variational wave equation. J Differential Equations. 2015;259(1):172-200.
95. Hu Y. Conservative solutions to a nonlinear variational sine-Gordon equation. J Math Anal Appl. 2012;385(2):1055-1069.
96. Grunert K, Holden H. Uniqueness of conservative solutions for the Hunter-Saxton equation. Research in the Mathematical Sciences. 2022;9(2):30.
97. Holden H, Raynaud X. Global dissipative multipeakon solutions of the Camassa-Holm equation. Commun Partial Differential Equations. 2008;33(11):2040-2063.
98. Bressan A, Huang T. Representation of dissipative solutions to a nonlinear variational wave equation. Communications in Mathematical Sciences. 2016;14(1):31-53.
99. Cieślak T, Jamróz G. Maximal dissipation in Hunter-Saxton equation for bounded energy initial data. Adv Math. 2016;290:590-613.
100. Bressan A, Holden H, Raynaud X. Lipschitz metric for the Hunter-Saxton equation. Journal de Mathématiques Pures et Appliquées. 2010;94(1):68-92.
101. Carrillo JA, Grunert K, Holden H. A Lipschitz metric for the Hunter-Saxton equation. Commun Partial Differential Equations. 2019;44(4):309-334.
102. Grunert K, Tandy M. Lipschitz stability for the Hunter-Saxton equation. J Hyperbol Differ Equ. 2022;19(2):275-310.
103. Grunert K, Tandy M. A Lipschitz metric for alpha-dissipative solutions to the Hunter-Saxton equation. Partial Differential Equations and Applications. 2024;5(4):24.
104. Wei X. Global weak solutions for a periodic Hunter-Saxton equation with weak dissipation. International Journal of Mathematics. 2012;23(4):1250036.
105. Wei X. Global weak solution for a periodic generalized Hunter-Saxton equation. J Math Anal Appl. 2012;391(2):530-543.
106. Liu J. Global weak solutions for the weakly dissipative mu-Hunter-Saxton equation. Ukrainian Mathematical Journal. 2014;65(8):1217-1230.
107. Wang F, Li F, Chen Q. On the Cauchy problem for a weakly dissipative generalized mu-Hunter-Saxton equation. Monatshefte für Mathematik. 2015;181(3):715-744.
108. Arbabi S, Nazari A, Darvishi MT. A semi-analytical solution of Hunter-Saxton equation. Optik. 2016;127(13):5255-5258.
109. Parand K, Delkhosh M. An efficient numerical solution of nonlinear Hunter-Saxton equation. Communications in Theoretical Physics. 2017;67(5):483-492.
110. Hashmi MS, Awais M, Waheed A, Ali Q. Numerical treatment of Hunter Saxton equation using cubic trigonometric B-spline collocation method. AIP Advances. 2017;7(9):095124.
111. Nirmala AN, Kumbinarasaiah S. Numerical solution of nonlinear Hunter-Saxton equation, Benjamin-Bona Mahony equation, and Klein-Gordon equation using Hosoya polynomial method. Results in Control and Optimization. 2024;14:100388.
112. Ivanov RI. Algebraic discretization of the Camassa-Holm and Hunter-Saxton equations. J Nonlinear Math Phys. 2008;15(S2):1-10.
113. Christiansen T. On the convergence rate of a numerical method for the Hunter-Saxton equation. IMA Journal of Numerical Analysis. 2026.
114. Camassa R, Huang J, Lee L. On a completely integrable numerical scheme for a nonlinear shallow-water wave equation. J Nonlinear Math Phys. 2005;12(S1):146-166.
115. Camassa R, Huang J, Lee L. Integral and integrable algorithms for a nonlinear shallow-water wave equation. J Comput Phys. 2006;216(2):547-572.
116. Camassa R, Lee L. Complete integrable particle methods and the recurrence of initial states for a nonlinear shallow-water wave equation. J Comput Phys. 2008;227(15):7206-7221.
117. Wunsch M. On the Hunter-Saxton system. Discrete and Continuous Dynamical Systems - B. 2009;12(3):647-656.
118. Wunsch M. The generalized Hunter-Saxton system. SIAM J Math Anal. 2010;42(3):1286-1304.
119. Wunsch M. Weak geodesic flow on a semidirect product and global solutions to the periodic Hunter-Saxton system. Nonlinear Anal. 2011;74(15):4951-4960.
120. Zuo D. A two-component mu-Hunter-Saxton equation. Inverse Problems. 2010;26(8):085003.
121. Hou Y, Fan E. Algebro-geometric solutions for the two-component Hunter-Saxton hierarchy. J Nonlinear Math Phys. 2014;21(4):473-484.
122. Li C, Wen S, Chen A. Single peak solitary wave and compacton solutions of the generalized two-component Hunter-Saxton system. Nonlinear Dynamics. 2014;79(2):1575-1585.
123. Maier L. On global weak solutions for the magnetic two-component Hunter-Saxton system. J Dyn Differ Equ. 2026.
124. Xie MM, Tian SF, Yan XJ. On the generalized Fourier transform for the modified Hunter-Saxton equation. J Geom Phys. 2025;217:105647.
125. Bayrakdar T, Ok Bayrakdar Z. A moving curve description of the Hunter-Saxton equation. Reports on Mathematical Physics. 2025;95(3):381-391.
126. Cotter C, Deasy J, Pryer T. The r-Hunter-Saxton equation, smooth and singular solutions and their approximation. Nonlinearity. 2020;33(10):5430-5465.
127. Zhang L, Hu B. The mixed Kuper-Camassa-Holm-Hunter-Saxton equations. J Nonlinear Math Phys. 2018;25(2):179-190.
128. Holden H, Karlsen KH, Pang PHC. The Hunter-Saxton equation with noise. J Differential Equations. 2021;270:725-786.
129. Podlubny I. Fractional differential equations. San Diego (CA): Academic Press; 1999. Mathematics in Science and Engineering; vol. 198.
130. Kilbas AA, Srivastava HM, Trujillo JJ. Theory and applications of fractional differential equations. Amsterdam: Elsevier; 2006. North-Holland Mathematics Studies; vol. 204.
131. Caputo M. Linear models of dissipation whose Q is almost frequency independent - II. Geophys J Int. 1967;13(5):529-539.
132. Caputo M, Mainardi F. A new dissipation model based on memory mechanism. Pure Appl Geophys. 1971;91(1):134-147.

133. Caputo M, Mainardi F. Linear models of dissipation in anelastic solids. La Rivista del Nuovo Cimento. 1971;1(2):161-198.
134. Haubold HJ, Mathai AM, Saxena RK. Mittag-Leffler functions and their applications. J Appl Math. 2011;2011:298628.
135. Gorenflo R, Kilbas AA, Mainardi F, Rogosin SV. Mittag-Leffler functions, related topics and applications. Berlin: Springer; 2014.
136. Atangana A, Baleanu D. New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model. Therm Sci. 2016;20(2):763-769.
137. Atangana A, Baleanu D, Alsaedi A. Analysis of time-fractional Hunter-Saxton equation: a model of neumatic liquid crystal. Open Physics. 2016;14(1):145-149.
138. Gómez-Aguilar JF, Atangana A. Fractional Hunter-Saxton equation involving partial operators with bi-order in Riemann-Liouville and Liouville-Caputo sense. Eur Phys J Plus. 2017;132(2):100.
139. Thomas R, Bakkyaraj T. A comparative study of Lie symmetry analysis and invariant subspace methods to fractional Hunter-Saxton equation. In: 2023 International Conference on Fractional Differentiation and Its Applications (ICFDA); 2023. p. 1-6.
140. Yu J, Feng Y. Lie symmetries, exact solutions and conservation laws of time fractional Hunter-Saxton system with variable coefficient. Engineering Computations. 2025;42(9):3543-3556.
141. Saha Ray S, Sahoo S. Traveling wave solutions to Riesz time-fractional Camassa-Holm equation in modeling for shallow-water waves. Journal of Computational and Nonlinear Dynamics. 2015;10(6):061026.
142. Gupta PK, Singh M, Yildirim A. Approximate analytical solution of the time-fractional Camassa-Holm, modified Camassa-Holm, and Degasperis-Procesi equations by homotopy perturbation method. Scientia Iranica. 2016;23(1):155-165.
143. Kumar D, Singh J, Baleanu D. A new analysis of the Fornberg-Whitham equation pertaining to a fractional derivative with Mittag-Leffler-type kernel. Eur Phys J Plus. 2018;133(2):70.