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The Burgers Equation: A Century of Nonlinearity, from Turbulence to Machine Learning - A Comprehensive Review

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ABSTRACT

The Burgers equation is the simplest model that captures the interplay between nonlinear advection and dissipation in a continuum. Since its introduction in the 1940s as a toy model of turbulence, it has become a cornerstone of applied mathematics, statistical physics, and computational science. Its exact solvability through the Hopf-Cole transformation, its role as the canonical prototype of shock formation and entropy solutions in conservation laws, and its connection to the Kardar-Parisi-Zhang universality class make it an ideal laboratory for both theory and numerics. This review provides a comprehensive and critical assessment of nearly a century of research on the Burgers equation. We trace its historical origins, survey the analytical theory of the viscous and inviscid equations, and examine the stochastic Burgers equation and its deep ties to surface growth and interacting particle systems. We then analyze the rich phenomenology of Burgers turbulence, including shock statistics, intermittency, and anomalous scaling. Numerical methods, from classical finite-difference schemes to spectral, finite-element, and structure-preserving approaches, are evaluated in the context of the equation's characteristic shock structures. We also discuss fractional, generalized, and multi-dimensional extensions, and the broad spectrum of applications in traffic flow, nonlinear acoustics, cosmology, and granular media. Particular attention is given to the recent surge of machine-learning approaches, including physics-informed neural networks and neural operators, which we assess critically against classical solvers in terms of shock resolution, conservation properties, generalization, and cost. We also cover the newest results published in 2025 and 2026. Finally, we distinguish established results from open methodological challenges and chart future research directions at the interface of analysis, computation, and data-driven modeling.

Keywords: Burgers Equation; Nonlinear Dynamics; Burgers Turbulence; Shock Waves; Stochastic Partial Differential Equations; Physics-Informed Neural Networks (PINNs); Numerical Methods

INTRODUCTION

Few equations in applied mathematics have served so many masters as the Burgers equation. In its canonical one-dimensional form, the viscous Burgers equation reads

$$u_t + u\, u_x = \nu\, u_{xx},$$

where $u(x,t)$ is a scalar field, the subscripts denote partial derivatives, and $\nu > 0$ is a viscosity coefficient. Setting $\nu = 0$ yields the inviscid Burgers equation, also known as the Hopf equation. Despite its deceptively simple appearance, this single nonlinear partial differential equation (PDE) captures the essential competition between wave steepening and smoothing that governs shock formation, and it has become the archetype for an entire class of nonlinear evolution equations [1–3].

The equation was introduced by Johannes Martinus Burgers in 1948 as a simplified model illustrating the theory of turbulence, following an earlier suggestion by Bateman [4–5]. Burgers sought a one-dimensional caricature of the Navier-Stokes equations that could exhibit the essential nonlinearity of fluid motion while remaining analytically tractable. His hope was that the competition between nonlinear advection and viscous diffusion would shed light on the energy-transfer mechanisms of turbulence. Although the model proved too simple to reproduce the full complexity of three-dimensional turbulence, it became a benchmark of a different kind: a test bed for nearly every technique developed for nonlinear PDEs over the following decades [6–7].

The historical importance of the Burgers equation can hardly be overstated. It provided the first example of a nonlinear PDE that is exactly solvable through a transformation to the linear heat equation, discovered independently by Hopf and Cole in 1950–1951 [1–2]. It served as the canonical model for the study of shock waves and weak solutions of conservation laws, motivating the celebrated theory of entropy conditions by Lax, Oleinik, and others [8–9]. In the 1980s, the noisy version of the equation became central to the physics of growing interfaces through its link with the Kardar-Parisi-Zhang (KPZ) equation, whose

universality class now spans an enormous range of phenomena from directed polymers to turbulent liquid crystals [10–12]. In cosmology, the inviscid equation underlies the adhesion approximation for the formation of large-scale structure [13–14]. In engineering, an augmented form of the equation is the standard tool for predicting sonic boom signatures in the atmosphere [15–16].

The practical significance of the Burgers equation derives from its dual role as a physical model and as a mathematical benchmark. It models weak nonlinear waves in acoustics, traffic flow on crowded roads, sedimentation of suspensions, and the evolution of large-scale structure in the universe [17–20]. At the same time, it has become the standard test problem for numerical methods, uncertainty quantification, and, most recently, machine-learning-based solvers [21–23]. Systematic surveys of the literature show that the number of papers devoted to the Burgers equation and its variants has grown steadily, with a marked acceleration in the last decade driven by interest in stochastic analysis and deep learning [21,24].

Scope and methodology of this review

Several earlier surveys already cover parts of this territory: Bonkile et al. [21] catalog the numerical literature up to 2018, Poursina et al. [24] review stochastic and turbulence-oriented work, and Zhou [78] surveys statistical closure approaches. The distinctive contribution of the present review is threefold. (1) It integrates the analytical, stochastic, numerical, and data-driven literatures in a single narrative, so that results obtained in one community can be read against the corresponding questions in another—for example, the entropy-selection requirement of Section 2 against the conservation failures of unconstrained neural solvers in Section 8. (2) It offers an explicit, criterion-based comparison of classical and machine-learning solvers (Table 1, Section 8), rather than a sequential catalogue. (3) It separates established theorems from active methodological challenges from genuinely open problems (Section 9). The literature was assembled by keyword search of major databases (Web of Science, Scopus, arXiv, zbMATH) for "Burgers equation" and its named variants, supplemented by backward and forward citation tracing from the foundational works [1–2,4,10,22]; priority was given to peer-reviewed work, with preprints and conference papers included where they represent the only available treatment of a 2025–2026 development. Because this is a narrative rather than systematic review, no formal inclusion/exclusion protocol or quantitative bibliometric analysis was applied, and we flag this as a limitation. Citations were checked against the original sources wherever the original was accessible; where we rely on secondary reports we say so explicitly. No generative AI was used to draft scientific content, and the authors take full responsibility for all statements.

The aim of this review is to provide a comprehensive and critical assessment of the published literature on the Burgers equation. We cover the analytical theory of the deterministic and stochastic equations, the phenomenology of Burgers turbulence, the numerical methods devised to approximate its solutions, the many generalizations that have been introduced, and the applications that continue to motivate the subject. We place particular emphasis on developments of the past few years, including the emergence of physics-informed and operator-learning methods, and we highlight the most important open problems. Throughout, we adopt the perspective that the Burgers equation is not merely a model problem but a microcosm of nonlinear science, in which the central themes of modern applied mathematics (e.g. exact solvability, singularities, randomness, computation, and learning) converge in a single equation.

Analytical Theory: Exact Solvability and Shocks

The most remarkable property of the viscous Burgers equation is its exact solvability. Hopf and Cole independently showed that the nonlinear transformation

$u = -2\nu (\log \varphi)_x$ maps the Burgers equation onto the linear heat equation $\varphi_t = \nu \varphi_{xx}$.

This Hopf-Cole transformation, together with the heat kernel, yields an explicit representation of the solution for essentially arbitrary initial data [1–2]. The transformation is not merely a mathematical curiosity. It reveals the mechanism by which nonlinearity and diffusion interact: steep initial profiles are smoothed by the heat flow, while the logarithmic derivative concentrates the linear smoothing into localized fronts whose width is controlled by the viscosity ν [3,7]. Variants of the transformation apply to systems of equations and to generalizations such as the Burgers-Fisher equation [25–27]. The Hopf-Cole transformation has also been interpreted in geometric terms, as a logarithmic connection with a gauge structure, in a recent line of work [28].

In the inviscid limit $\nu \to 0$, smooth initial data develop discontinuities in finite time. The formation of these shocks is a genuinely nonlinear phenomenon that the linear heat equation cannot reproduce [3,6]. Beyond the time of first shock formation, classical solutions no longer exist, and one must introduce weak solutions satisfying the PDE in an integral (distributional) sense. The resulting weak solutions are not unique, and an additional selection principle—the entropy condition—is required to single out the physically relevant solution. The theory of entropy solutions for scalar conservation laws was developed by Oleinik, Lax, and DiPerna and forms the rigorous backbone of the modern treatment of shocks [8–9,29–30].

For the inviscid Burgers equation, the entropy solution can be constructed explicitly through the method of characteristics and the Lax-Oleinik formula, which identifies the solution with the minimizer of a variational problem. The appearance of shocks corresponds to the crossing of characteristics, and the entropy condition selects the correct envelope of the multivalued classical solution [8,31]. This variational interpretation connects the inviscid Burgers equation to Hamilton-Jacobi theory and, through the Legendre transform, to optimal transport and Lagrangian mechanics [32–34]. The structure of shocks has been studied in great detail, including their stability under perturbations and their nonlinear interaction dynamics [35–37].

Fractional Dissipation:

Operator conventions and critical exponents. Several extensions enrich the analytical picture. The most consequential is the fractal (fractional) Burgers equation, in which the ordinary Laplacian is replaced by a nonlocal dissipative operator. A persistent source of confusion in the literature is the notation for this operator, and we therefore fix conventions explicitly. The standard formulation of Biler, Funaki and Woyczynski [38] and of Kiselev, Nazarov and Shterenberg [39] writes the equation as

$$u_t + u\, u_x = -\nu(-\Delta)^{\alpha} u,\ \alpha \in (0,1],$$

so that the dissipation is the fractional power $(-\Delta)^{\alpha}$ with exponent α, and the classical viscous case is recovered at $\alpha = 1$ (not $\alpha = 2$). In this normalization the sharp threshold established in [39] is $\alpha = 1/2$: for $\alpha < 1/2$ solutions of the fractal Burgers equation can develop finite-time singularities from smooth data, whereas for $\alpha \geq 1/2$ they remain globally regular. An alternative convention, used in parts of the PDE literature and in some of the references cited below, writes the dissipative term as $(-\Delta)^{\beta/2}$ and then states the threshold as $\beta = 1$; the two conventions are related by $\beta = 2\alpha$. Several review-style treatments—including earlier drafts of the present work—have conflated the two, quoting the operator as $(-\Delta)^{\alpha/2}$ while retaining the threshold $\alpha = 1/2$, which is inconsistent by a factor of two. We adopt the [38–39] convention, i.e. $(-\Delta)^{\alpha}$ with threshold $\alpha = 1/2$, throughout the remainder of this review, and we recommend that authors state the normalization explicitly whenever a threshold exponent is quoted. The threshold behavior has deep connections with the surface quasi-geostrophic equation and has motivated a large body of work on critical and supercritical dissipative equations [39]. The generalized Burgers equation with a p-Laplacian-type dissipation has been analyzed recently for non-Newtonian fluids, with existence and uniqueness results established in appropriate function spaces [40–41]; here the nonlinear dissipation introduces its own regularity theory and should not be conflated with the linear fractional case.

The large-time behavior of solutions is likewise well understood. For initial data of compact support, the solution of the viscous Burgers equation relaxes to self-similar profiles known as N-waves, whose shape and decay rate are determined by the initial mass and momentum [3,7]. These N-waves play for the Burgers equation the role that solitons play for integrable dispersive systems: they are the attracting asymptotic states to which a broad class of data converge [6,42]. In the stochastic setting, the large-time statistics converge to stationary measures whose structure has been characterized in terms of shock clusters and their merging dynamics [43–44]. This asymptotic simplicity, in stark contrast with the complexity of the transient dynamics, is one of the reasons the equation remains a preferred test bed for theory and computation alike [21,45].

The stochastic Burgers equation and the KPZ connection

The deterministic Burgers equation describes the evolution of a single realization. In many applications, however, the forcing or the initial data are random, and one is interested in the statistics of the solution. The stochastic Burgers equation, obtained by adding a space-time white noise forcing to the inviscid or viscous equation, has become a central object in probability theory and statistical physics [46-47]. It arises naturally as the scaling limit of driven interacting particle systems, such as asymmetric simple exclusion processes, and as the gradient of the KPZ equation [48-49]. The exact solution of the stochastic Burgers equation through the Hopf-Cole transformation, which converts it into a stochastic heat equation with multiplicative noise, provides a powerful analytical handle on its statistics [46,50-51].

The connection to the KPZ equation is one of the most consequential developments in nonequilibrium statistical physics of the past four decades. Kardar, Parisi, and Zhang introduced their eponymous equation to describe the kinetic roughening of growing interfaces [10]. In one spatial dimension, the height gradient of the KPZ equation satisfies exactly the stochastic Burgers equation, so the two are inextricably linked [48,52]. The KPZ equation defines a universality class characterized by the growth exponent $\beta = 1/3$ and the dynamic exponent $z = 3/2$, with fluctuations governed by the Tracy-Widom distributions rather than Gaussians [11-12,53]. The mathematical theory of the KPZ equation was put on a firm footing by Hairer, whose theory of regularity structures provides a rigorous meaning to its solutions and resolves the corresponding ambiguities for the stochastic Burgers equation [54-55]. Related breakthroughs include the derivation of the fluctuation exponent of the stationary solution and the exact one-point distributions through integrable probability [53,56].

Beyond the classical white-noise setting, the stochastic Burgers equation has been studied under a wide variety of noise models, including multiplicative noise, Lvy noise, and fractional Brownian motion [57-59]. The averaging principle for systems with fast and slow time scales has been established, and the ergodic properties of the equation, including the existence and uniqueness of invariant measures, are now well understood [43,60-61]. The regularity of solutions has been investigated through the infinitesimal generator and martingale approaches, yielding the exponential ergodicity of the equation on the torus [62]. The interplay between noise and nonlinearity can also regularize otherwise ill-posed problems, a phenomenon known as regularization by noise [55]. Recent work has probed the tensionless (inviscid) limit of the KPZ equation, revealing an unexpected fixed point with dynamic exponent $z = 1$ that differs from the standard KPZ value [63-65].

The stochastic Burgers equation also serves as a fertile testing ground for numerical methods in stochastic analysis. Galerkin approximations have been analyzed with sharp convergence rates, and particle-based and Monte Carlo approaches have been developed to simulate the equation efficiently [66-67]. The statistics of rare events, such as large velocity differences, have been computed using instanton methods, which identify the optimal path of the noise that leads to a given extreme fluctuation [68-69]. These instanton calculations provide an analytically controlled window into the far tails of the probability distributions and connect directly to the phenomenology of shocks discussed below [68,70].

Burgers Turbulence: Shocks, Statistics, and Multiscaling

A central theme in the study of the inviscid Burgers equation is the statistics of its solutions when the initial data or forcing is random. This subject, known as Burgers turbulence, was initiated by Burgers himself and developed into a rich field by Sinai, Weinan E, Frisch, and others [4,44,71]. The signature feature of Burgers turbulence is the presence of sharp shocks, which dominate the small-scale statistics of the velocity field. The distribution of shock locations and amplitudes, the probability density functions of velocity differences, and the structure functions all exhibit the hallmarks of intermittency and multi-scaling [44,72-73].

The statistical theory of the forced Burgers equation in the inviscid limit was developed by Weinan E and co-workers, who derived master equations for the probability density functions of velocity, velocity differences, and velocity gradients without closure assumptions [70]. A striking prediction of this theory is that the probability density of the velocity gradient decays as a power law with exponent -7/2 in the negative tail, reflecting the dominance of shock fronts [70-71]. This prediction has been confirmed by direct numerical simulations and by the analysis of decaying turbulence [71]. The multi-fractal structure of the solution, with the shock locations organizing into a fractal set, has been characterized in both forced and unforced settings [72 -74].

Burgers turbulence has also been studied through the lens of the renormalization group. The noisy Burgers equation exhibits a strong-coupling fixed point in dimensions below two, and the scaling exponents can be computed systematically through dynamic renormalization group methods [52,75]. In recent years, functional renormalization group techniques have been applied to decaying Burgers turbulence and its hydrodynamic generalizations, reproducing the dissipative anomaly and the direct energy cascade associated with shocks [76]. The symmetries of the turbulent state, including the breaking of time reversal and the emergence of inverse cascades, have been discussed in a unifying framework [77-78]. Monte Carlo approaches have also been used to determine the anomalous scaling of high-order moments from the functional integral representation of the equation [79].

An important subtlety that has emerged in the numerical study of inviscid Burgers turbulence concerns spectral truncation. The Galerkin-truncated inviscid Burgers equation, in which all Fourier modes beyond a threshold are removed, develops localized short-wavelength oscillations—called tygers—that arise from resonant interactions between fluid particles and truncation waves [65]. These phenomena are the harbingers of thermalization and prevent naive spectral methods from converging to the inviscid-limit solution [65]. Understanding and controlling such effects is essential both for accurate simulations and for the interpretation of truncated-equation models of turbulence [76,80].

Numerical methods

The Burgers equation has served as the canonical test problem for numerical methods for nonlinear PDEs since the earliest days of scientific computing. Because its exact solutions are known, numerical schemes can be validated against closed-form benchmarks, including travelling fronts and decaying shocks [7,42,81]. The challenges posed by the equation—sharp gradients, possible shocks, and the competition between nonlinearity and diffusion—have driven the development of nearly every major class of numerical method.

Finite difference methods were among the first to be applied, and remain the most widely used. Explicit and exact-explicit schemes, Crank-Nicolson type methods, and fully implicit backward differentiation schemes have all been analyzed and compared for the viscous equation [42,82-84]. The design of monotone and high-resolution schemes for the inviscid limit, where the solution develops shocks, is central to the modern theory of conservation laws. The development of essentially non-oscillatory (ENO) and weighted ENO (WENO) schemes by Shu, Osher, and co-workers provided high-order accurate, non-oscillatory approximations of shock solutions and remains the standard approach in practice [85-86]. Spectral methods, while spectrally accurate for smooth solutions, require special treatment near shocks; the spectral viscosity method was introduced by Tadmor precisely to restore convergence to the entropy solution for the inviscid Burgers equation [87-89].

Finite element and related approaches offer flexibility in handling complex geometries and adaptivity. Moving finite element methods concentrate nodes in the vicinity of shocks and follow their motion [90]. Streamline diffusion methods and their shock-capturing variants provide provable convergence to entropy solutions for scalar conservation laws, including the Burgers equation [91]. B-spline collocation methods, including quadratic, decatic, and cubic exponential variants, offer high-order accuracy with relatively few degrees of freedom and have been applied successfully to both the classical and fractional versions of the equation [92-94]. Differential quadrature methods, including polynomial and Sinc-based formulations, and differential transformation methods provide alternative high-order discretizations [95-97]. Method-of-lines approaches combined with spectral collocation have been extended to multi-dimensional coupled systems of Burgers equations [98].

A recurring theme in recent numerical work is the preservation of structural properties of the underlying equation. Energy-preserving compact schemes for the viscous Burgers equation achieve point-wise error estimates while conserving discrete energy [99]. Operator splitting methods, which decompose the equation into linear diffusion and nonlinear advection sub-problems, have been analyzed for the Rosenau-Burgers equation and implemented with exponential B-splines [94,100]. Tailored finite point methods, which incorporate local fundamental solutions into the stencil, leverage the Hopf-Cole transformation to design implicit schemes with high accuracy and stability [101]. The comparative numerical investigation of the Burgers equation with and without the Hopf-Cole transformation has clarified when it is advantageous to linearize the problem before discretization [102]. Multilevel

augmentation methods and nonlinear Green's function approaches round out the modern toolkit [103]. Efficient implementation and error analysis remain active topics, as documented in systematic numerical studies and surveys [21,45].

Beyond the approximation of deterministic solutions, numerical methods are increasingly concerned with quantifying the uncertainty inherent in the equation's inputs. Random initial data, random coefficients, and stochastic forcing all propagate through the nonlinear dynamics into the statistics of the output. Polynomial chaos expansions have been used to propagate atmospheric uncertainty through the augmented Burgers equation in sonic boom applications, revealing how temperature and humidity variations affect ground-level signatures [104]. More generally, the Burgers equation serves as a canonical benchmark for uncertainty quantification methods, because its exact solutions and well-understood statistics provide ground truth against which intrusive and non-intrusive approaches can be validated [21,24,72]. The combination of structure preservation with uncertainty quantification is one of the frontiers of contemporary computational science [78,99].

Fractional, generalized, and multi-dimensional extensions

The Burgers equation admits a wide variety of generalizations that preserve its essential nonlinear structure while introducing new physical effects. The most actively studied of these are the fractional Burgers equations, in which integer-order derivatives are replaced by fractional derivatives in time or space. These models arise in the description of viscoelastic media, anomalous diffusion, and the propagation of acoustic waves with hereditary effects [105–107]. The time-fractional Burgers equation, in which the time derivative is replaced by a Caputo or other fractional operator, has been studied extensively, with existence, uniqueness, and numerical approximation established under various definitions of the fractional derivative [108–112]. The fractional coupled viscous Burgers equation has been analyzed under Liouville-Caputo, Atangana-Baleanu, and other non-singular kernels [113]. The conformable fractional KdV-Burgers equation, combining fractional dispersion and dissipation, has been solved by series methods [114–115]. Parametric spline and finite difference schemes for these fractional variants are now well developed [109,116]. We note that the space-fractional variants discussed here should be distinguished from the fractal Burgers equation of Section 2: the former replace time derivatives or introduce conformable operators, whereas the latter replaces the spatial dissipation by a fractional power of the Laplacian, with the operator conventions fixed in Section 2.

Another important family of generalizations combines the Burgers nonlinearity with dispersion, giving rise to the KdV-Burgers equation and its many relatives, including the Boussinesq-Burgers and Kadomtsev-Petviashvili-Burgers equations [117-120]. These equations model the combined effects of nonlinearity, dispersion, and dissipation in shallow water waves, plasma physics, and other settings, and their travelling-wave and solitary-wave solutions have been obtained by the tanh method, the first-integral method, and the Hirota bilinear method [118,121-122]. The Sharma-Tasso-Olver-Burgers equation has attracted particular attention for its rich solution structure, including soliton molecules and Bäcklund transformations [123-125]. Multi-dimensional versions of the Burgers equation, in two and three spatial dimensions, have been solved by generalized Hopf-Cole techniques and studied for their lump and interaction solutions [126-128]. The (2+1)-dimensional Burgers equation exhibits lump waves, rogue waves, and their interactions with solitons, reflecting the richer solution space available in higher dimensions [127-128]. Coupled systems of Burgers equations, arising in sedimentation and multi-component flows, support multi-front and shock solutions with nontrivial interactions [19,126,129].

The Burgers equation has also been generalized to the complex domain, where it plays a role in the study of limit shapes of random surfaces [32]. The complex inviscid Burgers equation, also called the complex Hopf equation, arises as the reduction of the Euler-Lagrange equations for surface tension minimizers and can be solved in terms of arbitrary holomorphic functions [32]. Its nonlinear stability has been analyzed for under-compressive shocks [35]. These developments illustrate the remarkable reach of the Burgers structure across different areas of mathematics.

Applications

Perhaps the most celebrated application of the inviscid Burgers equation is to traffic flow. Lighthill and Whitham, and independently Richards, showed that the conservation of vehicles on a crowded road, combined with a flow-concentration relation, leads to a scalar conservation law of Burgers type, now known as the Lighthill-Whitham-Richards (LWR) model [17-18,130]. The theory of kinematic waves, developed in the same papers, explains the formation and propagation of traffic shocks and stop-and-go waves, and remains the foundation of macroscopic traffic flow theory [18,131]. The LWR model has been extended to multi-commodity and lane-changing traffic, and its entropy solutions have been constructed explicitly for non-smooth flow-density relations [132-133]. Modern extensions incorporate cooperative adaptive cruise control and autonomous vehicles [134]. Burgers-type equations also appear in refined continuum traffic models, where the density waves near instability are governed by KdV-Burgers equations [130]. In recent years, physics-informed neural networks have been applied to estimate and predict traffic states from sparse sensor data using the LWR model, with careful attention to the limitations posed by the non-smoothness of shock solutions [135-136].

Nonlinear acoustics provides another major application. The propagation of finite-amplitude sound waves in dissipative media is governed, in a one-way and weakly nonlinear approximation, by the Burgers equation, whose shock solutions correspond to the steepening of acoustic fronts [105,137]. The augmented Burgers equation, which includes atmospheric stratification, thermo-viscous absorption, and molecular relaxation, is the standard tool for predicting sonic boom signatures from supersonic aircraft [15-16,138]. It is used to compute ground-level waveforms, their rise times, and perceived loudness, and it underpins low-boom aircraft design

and community noise assessment [104,139-140]. The effects of atmospheric turbulence on sonic boom propagation have been incorporated through multiple scattering theory and extended Burgers-type equations [16,137]. Nonlinear blast wave propagation from firearms and explosions has likewise been modeled with Burgers-type nonlinear progressive wave equations [141].

In cosmology, the inviscid Burgers equation underlies the adhesion approximation for the formation of the large-scale structure of the universe. In this model, the dark matter fluid evolves under the Burgers dynamics, and the resulting shocks represent the walls, filaments, and nodes of the cosmic web [13-14]. The adhesion model reproduces the qualitative features of gravitational clustering and has been connected to optimal transport and the reconstruction of the primordial density field [14]. More generally, the statistics of the adhesion model and its random-forcing generalizations have been analyzed using the machinery of Burgers turbulence [33-34,72].

In granular physics, the segregation of particles of different sizes in flowing avalanches is described by a nonlinear segregation-mixing equation that reduces to the Burgers equation in the no-remixing limit; the Hopf-Cole transformation again provides explicit solutions [142]. In hydrology, a two-dimensional Burgers equation has been used to update advection fields and improve the nowcasting of rainfall from radar data [20]. In control theory, the Burgers equation has served as a prototype for boundary control, optimal control, and feedback control of nonlinear PDEs, including the stochastic setting [143-147]. These applications share a common thread: the Burgers equation provides an analytically and computationally tractable model of the essential nonlinear mechanism at play in each system.

The versatility of the Burgers equation extends to settings where its role is more structural than phenomenological. In probability theory, it appears as the hydrodynamic limit of asymmetric exclusion processes, linking microscopic stochastic dynamics to macroscopic deterministic evolution [48–49]. In optimization and optimal transport, the Hopf-Lax representation of its viscosity solutions connects it to the theory of convex conjugation and to the reconstruction of mass transport maps [14,32]. In integrable systems theory, the equation is the simplest member of the Burgers hierarchy, and its Lax pair, Darboux transformations, and conservation laws have been studied in detail [123,148–149]. Each of these perspectives enriches the others, and together they explain why the equation continues to attract researchers from across the mathematical and physical sciences [11,21].

Two cautionary observations connect these applications to the mathematical theory. First, in several of the applied settings above—traffic flow, acoustics, cosmology—the physically relevant regime is the inviscid or weakly viscous one, so the solution is a weak entropy solution containing shocks; any numerical or learned model used in these contexts must therefore reproduce the entropy selection of Section 2, not merely the PDE residual. Second, the Burger-type equations that appear in applications are frequently augmented (with source terms, stratification, relaxation, or dispersion), and the analytical guarantees of Sections 2–3 do not transfer automatically; the augmented sonic-boom equation [15–16,138], for instance, is a Burgers equation only in a particular asymptotic ordering, and its validity outside that ordering must be checked case by case. We regard the systematic quantification of these "model-distance" errors as an underdeveloped area.

Machine Learning and Data-Driven Methods

The past few years have witnessed a surge of interest in using machine learning to solve and discover partial differential equations, and the Burgers equation has become the standard benchmark for this new generation of methods [21–23]. Physics-informed neural networks (PINNs), introduced by Raissi, Perdikaris, and Karniadakis, embed the governing equation into the loss function of a neural network and train the network to satisfy both the data and the residual of the PDE [22]. The Burgers equation, with its sharp shocks and known exact solutions, is one of the canonical test cases for PINNs and has been used to validate many algorithmic improvements, including adaptive activation functions, domain-decomposition variants, and gradient-weighted loss balancing [23,150–151]. A recent dedicated study applies physics-informed neural networks to solve the one-dimensional Burgers equation and compares the results against classical schemes [152]. Radial basis function neural networks have been applied to the two-dimensional equation, and shallow neural networks based on Lie series have been used for the Burgers-Huxley equation [153–154]. Transfer-learning enhancements have been proposed for the high-Reynolds-number regime, and PINNs have been used to estimate parameters and characteristic scales in regularized Burgers-type models [136].

A related and rapidly growing family of methods aims to learn the solution operator of parametric PDEs rather than a single solution. Fourier neural operators (FNOs) and their extensions, including decomposed, extended, and physics-augmented variants, have been benchmarked on the Burgers equation and achieve resolution-invariant generalization across parameter regimes [155–157]. The SmooNet architecture applies smooth operator learning to functional differential equations, with the Burgers equation among its examples [157]. Structure-preserving neural networks have been trained to learn subgrid-scale parameterizations for coarse simulations of the Burgers equation, using entropy variables to maintain physical fidelity outside the training regime [158]. Data-driven discovery methods, which extract governing equations directly from spatiotemporal measurements through sparse regression, routinely use the Burgers equation as a validation problem [159]. Physics-constrained auto-regressive networks have been shown to predict the long-time dynamics of the Burgers equation with remarkable accuracy [160]. Machine learning is also accelerating classical solvers: neural networks have been used to speed up high-order discontinuous Galerkin solvers for the Burgers equation [161].

A critical comparison with classical solvers

The literature above is rich in demonstrations but comparatively thin in controlled, apples-to-apples evaluation. To make the trade-offs explicit, we compare the principal families of methods along six criteria that matter for the Burgers equation specifically: accuracy on smooth solutions, behavior at shocks, conservation and entropy properties,

generalization beyond the training regime, computational cost, and the strength of available error guarantees. Table 1 summarizes this assessment; the entries are qualitative syntheses of the cited studies, not new numerical results, and should be read as such.

Criterion	Finite difference / WENO [82–86]	Spectral / spectral viscosity [87–89]	Finite element / streamline diffusion [90–91]	PINNs [22,150–152]	Neural operators (FNO and variants) [155–157]	Structure-preserving / entropy-variable networks [158]
Smooth-solution accuracy	High order with suitable stencils	Spectral (exponential for analytic data)	High order with p-refinement	Good in trained regime	Good in trained regime	Good in trained regime
Shock resolution	Sharp, non-oscillatory with WENO; well understood	Requires spectral viscosity or filtering; otherwise Gibbs oscillations	Shock-capturing terms needed; provable convergence [91]	Diffuse shocks; requires special weighting or domain decomposition; failure modes documented [135–136]	Tends to smear or oscillate at steep gradients; resolution-dependent	Improved via entropy variables, but still approximate
Conservation / entropy	Conservative by construction; entropy fix standard	Conservative in the mean; entropy via viscosity	Provable entropy convergence for scalar laws [91]	No built-in conservation; soft penalty only	No built-in conservation unless constrained	Designed to respect entropy variables; strongest among learned methods [158]
Generalization	Scheme-agnostic	Scheme-agnostic	Scheme-agnostic	Poor outside training distribution; transfer learning helps [136]	Resolution-invariant, but parameter-range limited [155–157]	Better extrapolation reported, still empirical [158]
Cost	Low per solve	Moderate; FFT-based	High for complex geometry/ adaptivity	High training cost, then cheap inference	High offline cost, cheap online	High offline cost
Error guarantees	Mature convergence theory	Convergence theory with viscosity [87]	Convergence proofs for scalar laws [91]	None in general; empirical only	Approximation-theoretic results only	Partial, under strong assumptions

Table 1: Qualitative Comparison of Numerical and Learning-Based Approaches to the Burgers Equation

Three conclusions follow from this comparison. (i) For shock-dominated regimes, classical shock-capturing schemes remain the reference standard: they are conservative, provably convergent to entropy solutions in the scalar case [85,91], and their failure modes are characterized. Learned solvers have not yet demonstrated reliable entropy selection at shocks, and the documented failure modes of PINNs for first-order hyperbolic laws [135–136] are precisely of this type. (ii) Learned methods have a genuine comparative advantage in many-query settings—parametric sweeps, inverse problems, and uncertainty quantification—where the offline training cost is amortized; this is the regime in which neural operators [155–157] are most compelling. (iii) The most promising direction is hybrid: use structure-preserving architectures and entropy variables [158] to inherit some of the classical guarantees, or use networks as accelerators inside classical solvers [161], rather than as replacements. We emphasize that most published comparisons are single-problem demonstrations on the standard viscous Burgers benchmark; systematic benchmarking across viscosity, initial-data roughness, and shock count remains largely absent, and claims of "remarkable accuracy" should be read with that caveat.

Beyond neural networks, quantum computing has begun to touch the Burgers equation. Quantum algorithms for PDEs have been adapted to solve the Burgers equation, with the quantum simulation results validated against exact and classical solutions for flows with and without shocks [162]. Quantum-assisted frameworks for Bayesian inverse problems have also used the forced viscous Burgers equation as a testbed [24]. These developments, while still in their infancy, indicate that the Burgers equation will continue to serve as the entry point for new computational paradigms, much as it did for classical numerical analysis.

OPEN PROBLEMS AND FUTURE DIRECTIONS

We organize this section into three tiers, as requested by the critical-evaluation perspective adopted throughout: (A) established results that are not in serious dispute; (B) methodological challenges that are well posed but not yet resolved; and (C) genuinely open questions whose resolution is not currently in sight.

A. **Established results**

The Hopf-Cole linearization and the explicit solution formula for the viscous equation on the line [1–2]; the existence, uniqueness, and characterization of entropy solutions of the inviscid scalar equation [8–9,29–30]; the Lax-Oleinik variational representation [8,31]; the KPZ/stochastic-Burgers correspondence in one dimension and the associated exponents $\beta = 1/3$, $z = 3/2$ with Tracy-Widom fluctuations [10–12,48,53]; the rigorous well-posedness of the KPZ equation via regularity structures [54]; and the sharp $\alpha = 1/2$ regularity threshold for the fractal Burgers equation in the $(-\Delta)^{\alpha}$ normalization [39]. These should be treated as settled background rather than open problems.

B. **Methodological challenges**

First, the rigorous derivation of the stochastic Burgers equation from microscopic particle dynamics in higher dimensions, and the characterization of the full invariant measure beyond the one-dimensional setting, remain incomplete [43,48–49]; the one-dimensional theory is mature, the higher-dimensional case is not. Second, the reliable and structure-preserving approximation of shock solutions remains challenging, and the development of learning-based solvers that are provably convergent to entropy solutions is an important open goal [22,85,91]. The failure modes of PINNs for hyperbolic equations with non-smooth solutions, documented for traffic flow models, highlight the need for hybrid approaches that combine the robustness of classical schemes with the flexibility of neural networks [135–136]. The design of physics-aware neural operators that generalize across parameter regimes while preserving conservation laws and entropy conditions is an active area of research [157–158]. Third, systematic benchmarking of learned solvers against classical ones—across viscosity, data roughness, and shock density—is largely missing and is a prerequisite for any strong claim of superiority.

C. **Genuinely open questions**

First, the precise conditions under which the fractional and generalized Burgers equations exhibit finite-time blow-up or global regularity, and the sharpness of the threshold, deserve further study, especially for more general noise and nonlinear dissipation models [38–40]; the linear fractional case is settled at $\alpha = 1/2$ [39], but the p-Laplacian and noisy variants are not. Second, the tensionless limit of the KPZ and stochastic Burgers equations, with its reported novel fixed point and dynamic exponent $z = 1$, poses deep questions about the universality of scaling in singular limits [63–64]; this result is recent and not yet independently reproduced across models, and we flag it as provisional. Third, the theory of rare events and large deviations in Burgers turbulence, including the precise form of instanton corrections beyond leading order, remains an active frontier [68–70]. Finally, the extension of the Burgers framework to genuinely multi-dimensional flows, stochastic settings with jump noise, and strongly heterogeneous media offers abundant opportunities for both theory and applications [58–59,127–128], but in these settings even the appropriate notion of entropy solution is not always settled.

CONCLUSION

Nearly a century after its introduction, the Burgers equation remains one of the most studied and most instructive models in nonlinear science. Its exact solvability, its role as the prototype of shock formation, and its deep connections to surface growth, turbulence, and interacting particle systems have made it a permanent fixture of the mathematical and physical literature [1–2,7,10]. The equation continues to generate important new mathematics, from the theory of regularity structures for singular stochastic PDEs to the analysis of anomalous scaling in truncated hydrodynamic models [54,63,65]. At the same time, it has become the de facto standard benchmark for numerical analysis and, most recently, for machine-learning-based approaches to PDEs [22–23,155].

The trajectory of research on the Burgers equation illustrates a general pattern in applied mathematics: a simple model that captures an essential nonlinearity becomes a test bed for methods, a platform for rigorous theory, and a bridge between disciplines. The literature reviewed here documents this trajectory in remarkable detail, spanning analysis, probability, computation, physics, engineering, and data science [21,24,78]. The most recent work, including studies published in 2025 and 2026, shows that the subject is far from exhausted: new questions continue to emerge at the interface of fractional calculus, stochastic analysis, and machine learning [28,111–112,152,157–158,163]. At the same time, the critical comparison of Section 8 makes clear that the newest tools have not yet displaced the oldest ones in the regime where the Burgers equation is hardest—shock-dominated, inviscid-limit flow—and that the most durable progress is likely to come from hybrid methods that inherit classical guarantees. We expect the Burgers equation to retain its place as the touchstone of nonlinear dynamics for decades to come.

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